EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER1

TIME: Tuesday, March 18, 2025, 8–12

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 4

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VISITS: 9 and 11

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APPROVED TOOLS: Collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the exam.

The exam can be inspected and checked out 2025-04-10, 12.30-13.00, 2A:573, B-building, entrance 25, A-corridor, to the right.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) Find the control signal $u(t)$ expressed as a function of the states $x(t)$ which satisfies the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & x(T) + \int_0^T \frac{u^2(t)}{x(t)}\, dt \\ \text{subject to} \quad & \dot{x}(t) = -u(t), \end{aligned}$$

for a fixed T, using the PMP. (5p)

Hint: The adjoint equation is a separable ODE.

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \left(e^t y + \frac{\dot{y}^2}{t} \right) dt,$$

satisfying $y(0) = 1$ and $y(1) = 0$. (5p)

2. In a simple model of economic growth the following varaibles are considered.

- x. Capital.
- u. Consumption.
- $I = \dot{x}$. Investment.
- $y = f(x)$. The economic output.

The economic output can either be invested or consumed so

$$y = \dot{x} + u$$

It is assumed that the utility to society of the consumption at time t is given by a function

$$U(u)e^{-\rho t}$$

An optimal growth policy would then try to maximize

$$J = \int_0^T U(u)e^{-\rho t} dt$$

over some time interval $[0, T]$. Below we will assume that

$$U(u) = \ln u, \quad f(x) = x^{1/3}$$

(Note that u and x are positive quantities.)

(a) Use the Pontryagin minimum principle to derive a set of differential equations that are necessary conditions for an optimal growth policy u. Assume that the boundary conditions are $x(0) = x_o$ and $x(T) = x_T$. **Note:** The differential equations only have to be formulated, not solved. (5p)

(b) Assume that $x_o = x_T$. Is it possible to choose x_o so that there is a constant solution $x(t) = x_o$ to the equations in (a)? What is this value? What is the corresponding control signal u? (5p)

3. We are interested in text justification which is about making lines in a text of N words have about the same length. More precisely we let the line length $l(i, j)$ be the characters of words i through j including counting blanks inbetween the words. We also let L be the maximum allowed length of a line. Then $f(i, j)$, which we call the badness of a line with words i through j, is given by

$$f(i,j) = (L - l(i,j))^3,\ 0 \le l(i,j) \le L$$

and $f(i, j) = \infty$ if $l(i, j) > L$. We are intested in the minimal overall badness defined as:

$$J = \min_{\text{partition of text}} \sum_{k=1}^{K} f(i_k, j_k)$$

Here the partition is defined such that $i_1 = 1$, $i_{k+1} = j_k + 1$ and $j_K = N$ for some K, where K is the number of lines. Notice that we do not know beforehand how many lines there will be, and that this will come out of the optimal solution. Introduce also the minimal badness if starting with word i and considering the remaining part of the text:

$$J(i) = \min_{\text{partion of text from word } i} \sum_{k=1}^{K_i} f(i_k, j_k)$$

Then it can be shown that

$$J(i) = \min_{i \le j \le N} \left(f(i,j) + J(j+1) \right)$$

with $J(N + 1) = 0$. You should now consider the text "TO BE OR NOT TO BE". Your task is to compute the optimal partition for this text such that the overall badness is minimized when $L = 6$. What is the resulting overall badness? Moreover, redo your calculations when the text instead is "BE OR NOT TO BE". (10p)

4. We consider a one degree of freedom robot joint with constant joint stiffness K and damping D. The dynamics is described by

$$M\frac{d^2q}{dt^2} + D\frac{dq}{dt} + Kq = K\theta + D\frac{d\theta}{dt}$$

where q is the link position, θ is the motor position, and M is the link inertia.

(a) Show that with $x_1 = \theta - q$, $x_2 = \dot{q}$ and $u = \dot{\theta}$ the dynamics can equivalently be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2 + u\\ \omega_0^2 x_1 + 2\zeta\omega_0(-x_2 + u)\end{bmatrix}$$

where $\omega_0^2 = K/M$ and $\zeta = D/(2\omega_0 M)$. (2p)

(b) We are interested in maximizing the final link velocity $\dot{q}(t_f)$, where the final time t_f is fixed. The control signal u is constrained such that $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in [0, t_f]$. Form the Hamiltonian for this optimal control problem and write down the adjoint equations. (2p)

(c) Show that the optimal control is given by

$$u^*(t) = \begin{cases} \overline{u}(t),\ \sigma(t) < 0\\ \underline{u}(t),\ \sigma(t) > 0\\ \text{arbitrary},\ \sigma(t) = 0\end{cases}$$

where $\sigma(t) = \lambda_1(t) + 2\zeta\omega_0\lambda_2(t)$, and where $\lambda^T = \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}$ satisfies the adjoint equations. (2p)

(d) Consider the special case of $\zeta = 0$ which gives the switching function

$$\sigma(t) = -\omega_0 \sin(\omega(t_f - t)).$$

Discuss how many switches there will be. (2p)

(e) Consider the special case of $\zeta = 1$ which gives the switching function

$$\sigma(t) = \omega_0^2 e^{-\omega(t_f - t)}\left(t_f - t - \frac{2}{\omega_0}\right).$$

Discuss how many switches there will be. (2p)