TSRT08: Optimal Control
Solutions

20250318

1. (a) The Hamiltonian is given by

$$H(t,x,u,\lambda) = \frac{u^2}{x} + \lambda(-u).$$

Pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = \frac{2u}{x} - \lambda \quad \Rightarrow \quad u^* = \frac{1}{2}\lambda x.$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t,x,u,\lambda) = \frac{u^2}{x^2} = \frac{1}{4}\lambda^2,$$

which is a separable ODE in λ. We have

$$\frac{d\lambda}{dt} = \frac{1}{4}\lambda^2 \quad \Leftrightarrow \quad \frac{1}{\lambda^2}d\lambda = \frac{1}{4}dt \quad \Leftrightarrow \quad -\frac{1}{\lambda} = \frac{1}{4}t + c \quad \Leftrightarrow \quad \lambda = -\frac{4}{t+4c}$$

for some constant c. Since $\lambda(T) = \frac{\partial \phi}{\partial x} = 1$, we have $c = -1 - T/4$ and thus $\lambda = -4/(t - T - 4)$, which yields the optimal control signal

$$u^* = \frac{1}{2}\lambda x = -\frac{2}{t-T-4}x$$

(b) Introducing $x = y$ and $u = \dot{y}$, it holds that $\dot{x} = u$ and the Hamiltonian is given by

$$H(t,x,u,\lambda) = e^t x + u^2 t^{-1} + \lambda u.$$

The following equations must hold

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(t,x,u,\lambda) = -e^t, \tag{1a}$$

$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = 2ut^{-1} + \lambda. \tag{1b}$$

Now, (1a) yields $\lambda = -e^t + c_1$ and this inserted into (1b) yields

$$0 = 2ut^{-1} + \lambda = 2\dot{y}t^{-1} + c_1 - e^t \quad \Leftrightarrow \quad y = \frac{1}{2}\int t(e^t - c_1)\,dt = \frac{1}{2}(t-1)e^t - \frac{c_1}{4}t^2 + c_2.$$

Finally, $y(0) = 1$, $y(1) = 0$ gives the constants $c_1 = 6$ and $c_2 = 3/2$. Hence

$$y(t) = \frac{1}{2}\big((t-1)e^t - 3t^2 + 3\big),$$

is the sought after extrema.

2. (a) The optimal control problem can be stated as

$$\begin{aligned} \text{minimize} \quad & \int_0^T -\ln u(t) \cdot e^{-\rho t} dt \\ \text{subject to} \quad & \dot{x}(t) = x^{1/3}(t) - u(t) \end{aligned}$$

The Hamiltonian is given by

$$H(t, x, u, \lambda) = -\ln u \cdot e^{-\rho t} + \lambda(x^{1/3} - u)$$

Pointwise minimization yields

$$0 = \frac{\partial H}{\partial u} = -\frac{1}{u} e^{-\rho t} - \lambda \quad \Rightarrow \quad u = -\frac{e^{-\rho t}}{\lambda}$$

The Two Point Boundary Value Problem is then given by

$$\begin{aligned} \dot{\lambda}(t) &= -\frac{\partial H}{\partial x} = -\lambda(t) \frac{1}{3} x^{-2/3}(t) \\ \dot{x}(t) &= \frac{\partial H}{\partial \lambda} = x^{1/3}(t) - u(t) = x^{1/3}(t) + \frac{e^{-\rho t}}{\lambda(t)}, \qquad x(0) = x_0, \quad x(T) = x_T \end{aligned} \tag{2}$$

Note, since we have a constraint on $x(T)$, we will not get any constraint on $\lambda(T)$.

(b) Assume that $x(t) = x_0$ solves the the Two Point Boundary Value Problem. Since $\dot{x}(t) = 0$ under that assumption, we have

$$0 = \dot{x}(t) = x_0^{1/3} + \frac{e^{-\rho t}}{\lambda(t)} \quad \Rightarrow \quad \lambda(t) = -x_0^{-1/3} e^{-\rho t}$$

This result can be used in the adjoint equation

$$\dot{\lambda}(t) = -\lambda(t) \frac{1}{3} x^{-2/3}(t) \quad \Rightarrow \quad \rho x_0^{-1/3} e^{-\rho t} = x_0^{-1/3} e^{-\rho t} \frac{1}{3} x_0^{-2/3}$$

which gives the requirement

$$\rho = \frac{1}{3} x_0^{-2/3} \quad \Rightarrow \quad x_0 = (3\rho)^{-3/2}$$

The corresponding control signal will then be

$$\dot{x}(t) = x^{1/3}(t) - u^*(t) = 0 \Rightarrow u^*(t) = x_0^{1/3} = \frac{1}{\sqrt{3\rho}}$$

3. (a) TO BE OR NOT TO BE

In this case, the number of words N is 6, i.e. $N = 6$. Moreover, $L = 6$.

Stage $k = N + 1 = 7$: $J(7) = 0$.

Stage $k = 6$:

$$J(6) = \min_{6 \le j \le 6} \{f(6, j) + 0\} = f(6, 6) = 64 \tag{3}$$

Note that $f(6, 6) = (L - l(6, 6))^3 = (6 - l(6, 6))^3$. The last word is "BE", and the number of characters of this word is 2, therefore $l(6, 6) = 2$, and $f(6, 6) = 4^3 = 64$.

Stage $k = 5$:

$$\begin{aligned} J(5) &= \min_{5 \le j \le 6} \{f(5, j) + J(j + 1)\} \\ &= \min\{f(5, 5) + J(6), f(5, 6) + J(7)\} = \min\{128, 1\} = 1 \end{aligned} \tag{4}$$

Note that $f(5, 6) = (L - l(5, 6))^3 = (6 - l(5, 6))^3$. The fifth word is "TO", and the number of characters of this word through the sixth word ("BE") is $2 + 1 + 2$, therefore $l(5, 6) = 5$, and $f(5, 6) = (6 - 5)^3 = 1$.

optimum for (5,6)
Stage $k = 4$:

$$\begin{aligned} J(4) &= \min_{4\le j\le 6}\{f(4,j) + J(j+1)\} \\ &= \min_{4\le j\le 6}\{f(4,4) + J(5), f(4,5) + J(6), f(4,6) + J(7)\} \\ &= \min\{27+1, 0+64, \infty\} = 28 \end{aligned} \tag{5}$$

optimum for (4,4) and (5,6)
Stage $k = 3$:

$$\begin{aligned} J(3) &= \min_{3\le j\le 6}\{f(3,j) + J(j+1)\} \\ &= \min_{3\le j\le 6}\{f(3,3) + J(4), f(3,4) + J(5), f(3,5) + J(6), f(3,6) + J(7)\} = \\ &= \min\{64+28, 0+1, \infty, \infty\} = 1 \end{aligned} \tag{6}$$

optimum for (3,4) and (5,6)
Stage $k = 2$:

$$\begin{aligned} J(2) &= \min_{2\le j\le 6}\{f(2,j) + J(j+1)\} \\ &= \min_{2\le j\le 6}\{f(2,2) + J(3), f(2,3) + J(4), f(2,4) + J(5), f(2,5) + J(6), f(2,6) + J(7)\} \\ &= \min\{64+1, 1+28, \infty, \infty, \infty\} = 29 \end{aligned} \tag{7}$$

optimum for (2,3), (4) and (5,6)
Stage $k = 1$:

$$\begin{aligned} J(1) &= \min_{1\le j\le 6}\{f(1,j) + J(j+1)\} \\ &= \min_{1\le j\le 6}\{f(1,1) + J(2), f(1,2) + J(3), f(1,3) + J(4), f(1,4) + J(5), f(1,5) + J(6), f(1,6) + J(7)\} \\ &= \min\{64+29, 1+1, \infty, \infty, \infty, \infty\} = 2 \end{aligned} \tag{8}$$

optimum for (1,2), (3,4) and (5,6)
The minimum overall badness,

$$J^*(1) = 2, \tag{9}$$

achieved for the optimal partition of (1,2), (3,4), and (5,6).

(b) BE OR NOT TO BE, $N = 5$, $L = 6$.
Using the principle of optimality it can be read from (7) that the minimum overall badness,

$$J^*(1) = 29, \tag{10}$$

is achieved for the optimal partition of (1,2), (3), and (4,5).

4. (a) By introducing $x_1 = \theta - q$, $x_2 = \dot{q}$, $u = \dot{\theta}$ and redefining $x \triangleq (x_1, x_2)^T$, the problem at hand can be written on standard form as

$$\begin{aligned} \dot{x}_1(t) &= \dot{\theta} - \dot{q} = -x_2 + u, \\ \dot{x}_2(t) &= \ddot{q} = \frac{1}{M}\left(K(\theta - q) + D(\dot{\theta} - \dot{q})\right) = \frac{K}{M}x_1 + \frac{D}{M}\dot{x}_1 \end{aligned}$$

By introducing $\omega_0^2 = K/M$ and $\zeta = D/(2\omega_0 M)$ we have

$$\begin{aligned} \dot{x}_1 &= -x_2 + u, \\ \dot{x}_2 &= \omega_0^2 x_1 + 2\zeta\omega_0(-x_2 + u). \end{aligned}$$

(b) Considering $\phi(t_f, x(t_f)) = -x_2(t_f)$ and $f(t, x, u) = 0$, the Hamiltonian is given by

$$\begin{aligned} H(t, x, u, \lambda) &\triangleq f(t, x, u) + \lambda^T f(t, x, u) \\ &= \lambda_1(-x_2 + u) + \lambda_2(\omega_0^2 x_1 + 2\zeta\omega_0(-x_2 + u)). \end{aligned}$$

The adjoint equations are

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = -\lambda_2\omega_0^2 \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = \lambda_1 + 2\zeta\omega_0\lambda_2. \end{aligned}$$

(c) Pointwise minimization yields

$$\begin{aligned} \tilde{\mu}(t, x, \lambda) &\triangleq \underset{\underline{u} \le u \le \bar{u}}{\arg\min}\, H(t, x, u, \lambda) \\ &= \underset{\underline{u} \le u \le \bar{u}}{\arg\min}\, (\lambda_1 + 2\zeta\omega_0\lambda_2)u \\ &= \begin{cases} \bar{u}, & \sigma < 0 \\ \underline{u}, & \sigma > 0 \\ \tilde{u}, & \sigma = 0 \end{cases}, \end{aligned}$$

where $\tilde{u} \in [\underline{u}, \bar{u}]$ is arbitrary. Thus, the optimal control is expressed by

$$u^*(t) \triangleq \tilde{\mu}(t, x(t), \lambda(t)) = \begin{cases} \bar{u}, & \sigma(t) < 0 \\ \underline{u}, & \sigma(t) > 0 \\ \tilde{u}, & \sigma(t) = 0 \end{cases},$$

where the switching function is given by

$$\sigma(t) = \lambda_1(t) + 2\zeta\omega_0\lambda_2(t).$$

(d) Since the switching function never remains at zero for finite time, the optimal control, u^*, always takes its minimum or maximum values periodically, depending on the sign of $\sigma(t)$(known as bang-bang control). The number of times σ becomes zero, determines the number of switches, and $\sigma = 0$ when

$$\omega_0(t_f - t) = k\pi, k = 0, 1, 2, ... \Rightarrow t = t_f - \frac{k\pi}{\omega_0}$$

The number of switches is the largest integer, k, for which

$$t_f - \frac{k\pi}{\omega_0} > 0 \Rightarrow k < \frac{t_f\omega_0}{\pi}.$$

(e) The switching function crosses the time axis only once at $t' = t_f - \frac{2}{\omega_0}$, and switches sign from positive to negative. Therefore, if $t_f > \frac{2}{\omega_0}$, u^* starts from its minimum value and at t' switches to its maximum value. If $t_f \le \frac{2}{\omega_0}$, there will be no switches.