Optimal Control, Lecture 1: Introduction and Shortest Path

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Contents

1. Formulation of optimal control problems
2. Dynamic programming
3. Model predictive control
4. Reinforcement learning
5. Pontryagin maximum principle
6. Numerical methods

Applications



Applications



Applications

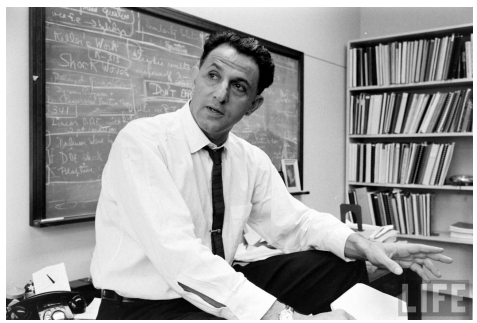


Applications

- Routing of airplanes
- Logistics
- Economical systems
- many more

History

- Roots in calculus of variations (Bernoulli, Euler, Lagrange, Weirstrass,...)
- Optimal control emerged in the 1950s during the space race
- Dynamic programming (Richard Bellman)
- Maximum principle (Lev Semenovich Pontryagin)
- Linear quadratic control (Rudolph Kalman)


Organization

- Examiner and lecturer: Anders Hansson, e-mail: anders.g.hansson@liu.se
- Course assistant: Reza Jafari, e-mail: seyyed.reza.jafari@liu.se (responsible for homeworks and exercise sessions)
- Course home page: http://www.control.isy.liu.se/student/tsrt08/

Course Evaluation and Course Development

- 21 registered students
- Evaliuate answer frequency 14.29%
- Overall evaluation of the course 4.67 (2x5, 1x4, 0x3, 0x2, 0x1)
- Criticism: Bra att matematiken respekteras och gås igenom korrekt. Jag är själv dock lite för svag matematiskt för att kunna förstå och uppskatta bevisen som presenterades på föreläsningarna. Lite väl högt tempo på tavelgenomgångarna då jag inte riktigt hann höra/ta till mig alla intressanta detaljer utan fick mest kämpa med att få ner engna anteckningar.
- Praise: Reza skötte lektionerna jättebra och nivån på lektionsuppgifterna var lagom utmanande. The lectures and labs have been fun!

Optimal Control

- Discrete time optimal control
- Continuous time optimal control

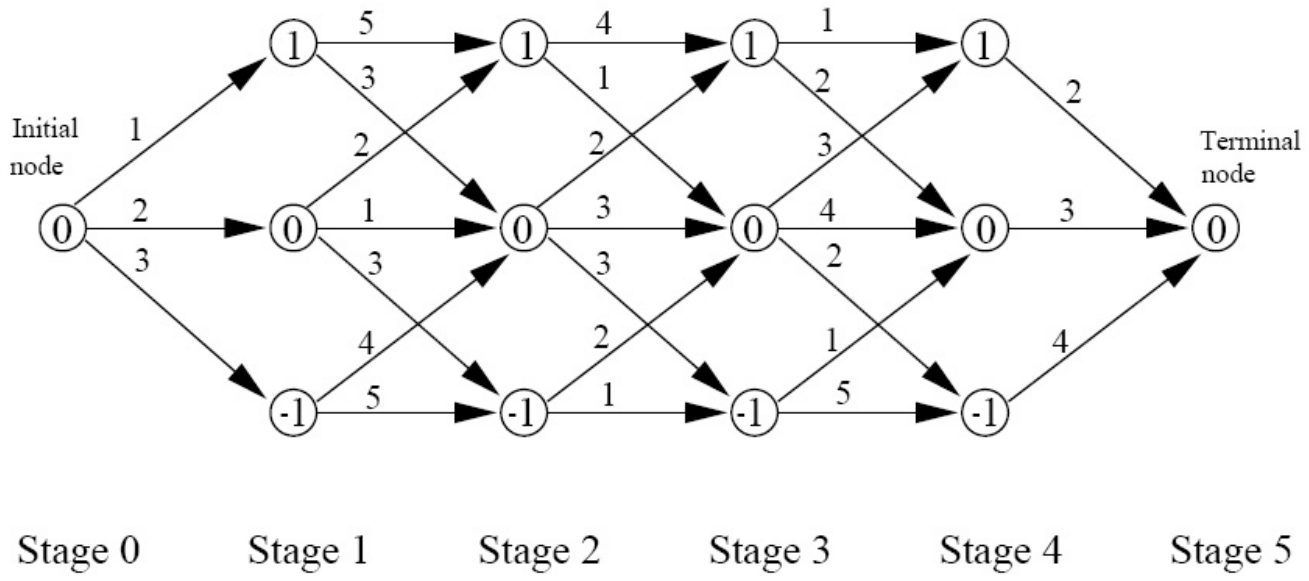
Discrete Time Optimal Control

$$
\begin{array}{ll}
\text{minimize} & \phi(x_N) + \sum_{k=0}^{N-1} f_k(x_k, u_k) \\
\text{subject to} & x_{k+1} = F_k(, x_k, u_k) \\
& x_0 \text{ given } , x_k \in X_k \\
& u_k \in U(k, x_k)
\end{array}
$$

Sequential decision problem which include as special cases

- Graph search problem
- Combinatorial optimization
- Discrete time optimal control

Shortest Path Problem



Find the shortest path from the initial node at stage 0 to the terminal node at stage 5.

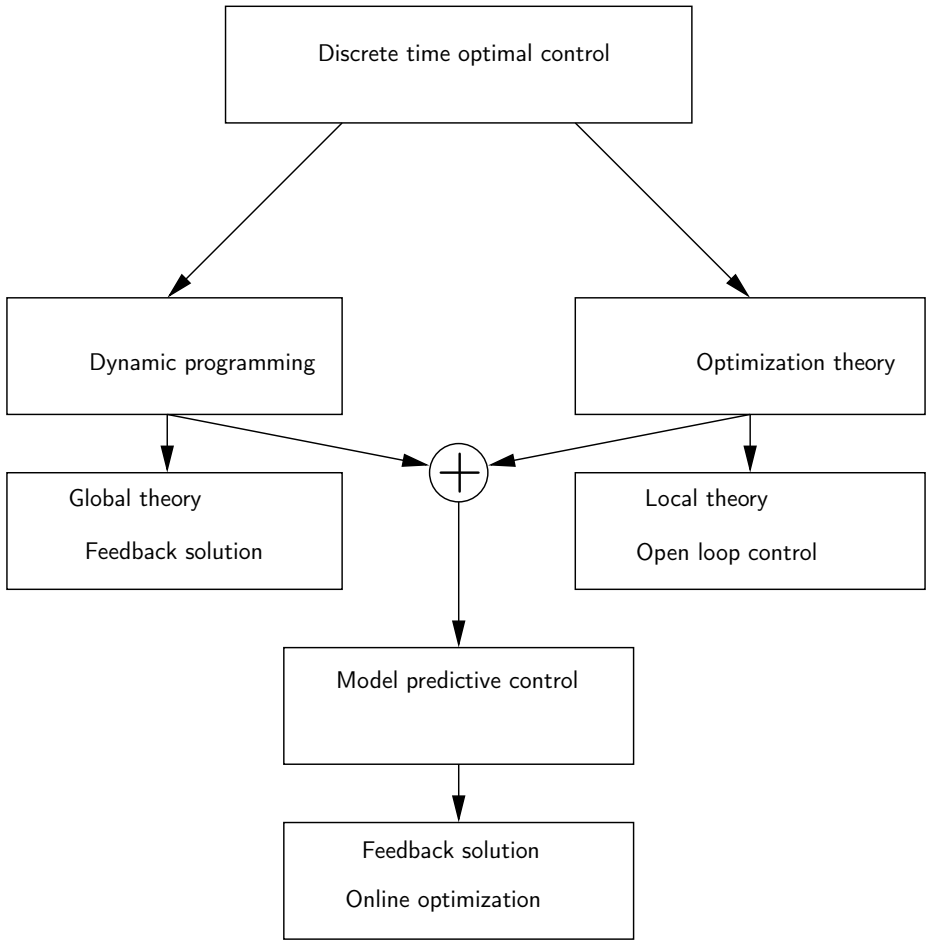
Knapsack Problem



maximize $\sum_{j=1}^{n} p_j x_j$

subject to $\sum_{j=1}^{n} w_j x_j \leq c, x_j = 0 \text{ or } 1, \ j = 1, \ldots, n$

Discrete Time Optimal Control



Reinforcement Learning

Is about solving the optimal control problem *without knowing* the dynamical model of the system to control.

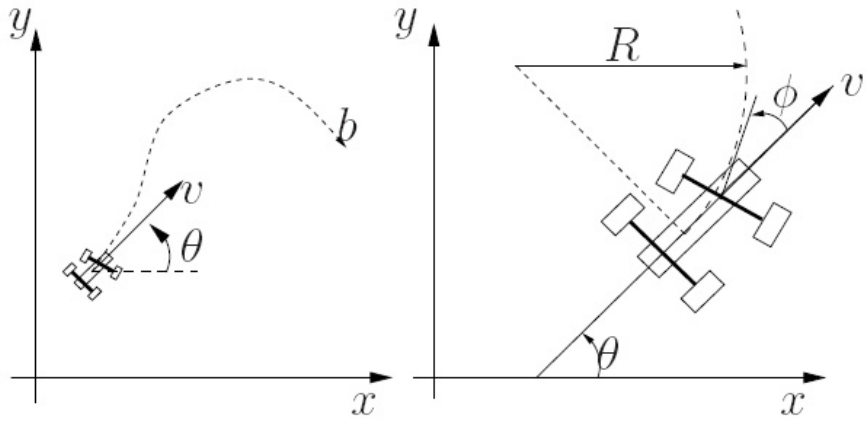
Based on carrying out experiments.

Continuous Time Optimal Control

$$
\begin{array}{ll}
\text{minimize} & \phi(t_1, x(t_1)) + \int_{t_0}^{t_1} f(t, x(t), u(t))dt \\
\text{subject to} & \dot{x}(t) = F(t, x(t), u(t)) \\
& x(t_0) \in S_0 \\
& x(t_1) \in S_1 \\
& u(t) \in U(x(t))
\end{array}
$$

S_0 and S_1 manifolds (intersections of surfaces)

Optimal Control of Car



Problem: shortest path (here t_1 is a free variable)

$$
\begin{array}{ll}
\text{minimize} & \int_0^{t_1} 1 dt \\
\text{subject to} & \dot{x}(t) = v\cos(\theta(t)),\ x(0) = 0,\ x(t_1) = \bar{x} \\
& \dot{y}(t) = v\sin(\theta(t)),\ y(0) = 0,\ y(t_1) = \bar{y} \\
& \dot{\theta}(t) = \omega(t),\ \theta(0) = 0,\ \theta(t_1) = \bar{\theta} \\
& |\omega(t)| \leq v/R,\ t_1 \geq 0
\end{array}
$$

Resource Allocation

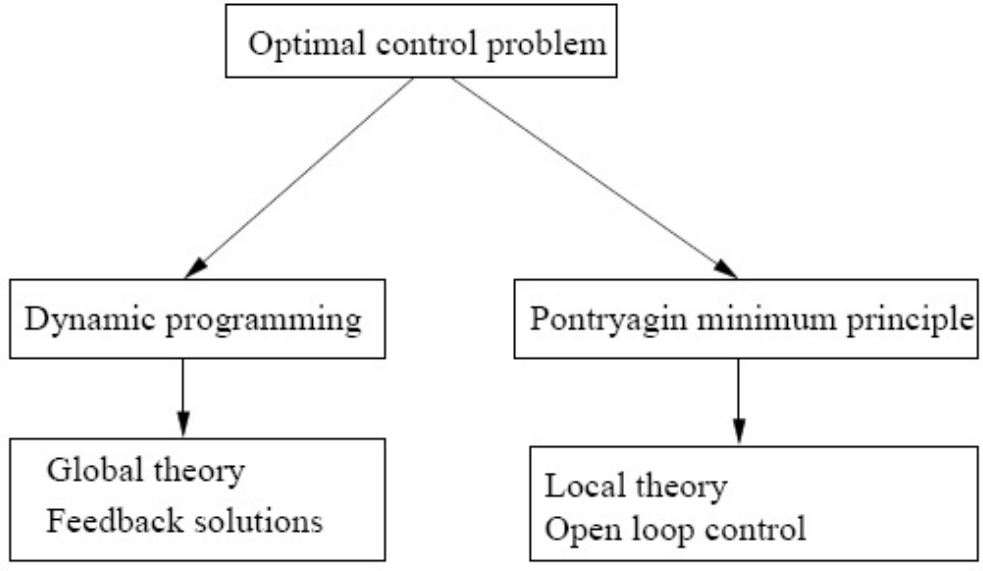
$$
\begin{array}{ll}
\text{maximize} & x(t_1) + \int_0^{t_1} (1 - u(t))x(t)dt \\
\text{subject to} & \dot{x}(t) = \alpha u(t)x(t),\ 0 < \alpha < 1 \\
& x(0) = x_0 > 0 \\
& u(t) \in [0, 1],\ \forall t
\end{array}
$$

The portion $u(t)$ of the production rate $x(t)$ is invested in the factory (and increases the production capacity)

The rest $(1 - u(t))x(t)$ is stored in a warehouse.

Maximize the sum of the total amount of goods stored and the final capacity.

Continuous Optimal Control



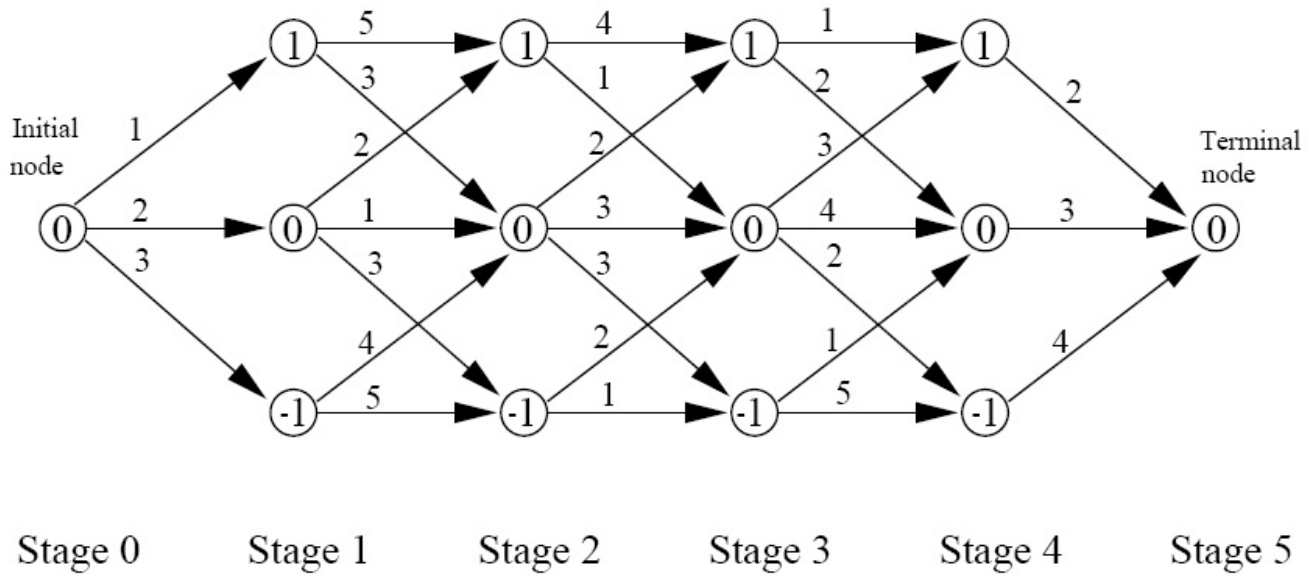
- Feedback control $u(t) = \psi(t, x(t))$ (depends on state)
- Open loop control $u(t) = \psi(t, x(t_0))$

Note: we will not do dynamic programming in continuous time.

Discrete Time Optimal Control

- Optimal control problems in discrete time
  - Discrete state space
  - Continuous state space
- In discrete time it is easier to understand
  - The principle of optimality
  - The dynamic programming equation
- Discrete version of the Pontryagin Maximum Principle (PMP)
- Optimal control on infinite time horizon
- Feedback versus open loop control
- Approximate solutions
- Model Predictive Control (MPC)
- Reinforcement learning

Shortest Path Problem



Find the shortest path from the initial node at stage 0 to the terminal node at stage 5.