Optimal Control, Lecture 2: Dynamic Programming

Anders Hansson

Division of Automatic Control
Linkoping University

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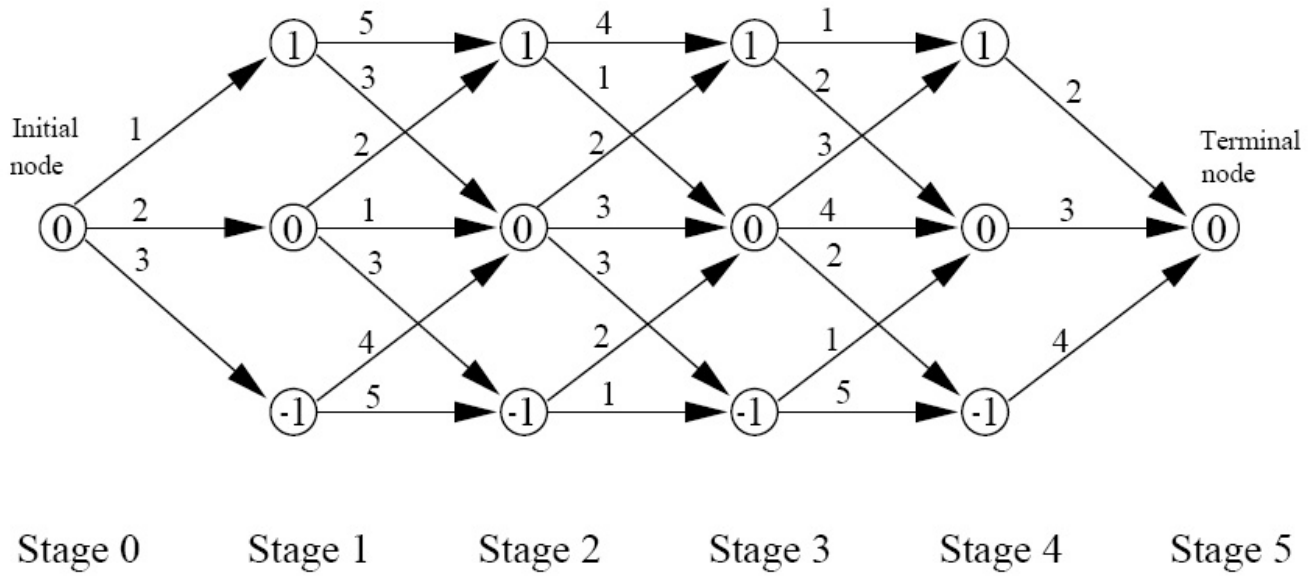
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Multistage Decision Problem

$$\begin{array}{ll} \text{minimize} & \phi(x_N) + \sum_{k=0}^{N-1} f_k(x_k, u_k) \\ \text{subject to} & x_{k+1} = F_k(x_k, u_k) \\ & x_0 \text{ given }, x_k \in X_k \\ & u_k \in U(k, x_k) \end{array} \tag{1}$$

k does not have to be time, but just enumeration of stages

Shortest Path Problem revisited



Find the shortest path from the initial node at stage 0 to the terminal node at stage 5.

and casted as Multistage Decision Problem

On white board

Principle of Optimality

If $\{u_k^*\}_{k=0}^{N-1}$ is optimal for (1), then $\{u_k^*\}_{k=n}^{N-1}$ is optimal for a problem on form (1) but with intial value (n, x_n^*) instead of $(0, x_0)$.

Optimal Cost-to-Go Function

$$\begin{array}{ll} J_l^*(x) = \text{min} & \phi(x_N) + \sum_{k=l}^{N-1} f_k(x_k, u_k) \\ \text{subject to} & x_{k+1} = F_k(x_k, u_k) \\ & x_l = x, x_k \in X_k \\ & u_k \in U_k(x_k) \end{array} \tag{2}$$

for $l = 0, 1, \ldots, N-1$ with $J_N^*(x) = \phi(x)$

Notice optimal value of (1) is $J_0^*(x_0)$

Dynamic Programming

Suppose there exist finite solution to the backward Dynamic Programming recursion

$$V_N(x) = \begin{cases} \phi(x), & x \in X_N \\ \infty, & x \notin X_N \end{cases}$$

$$V_k(x) = \min_{u \in U_k(x), F_k(x,u) \in X_{k+1}} \{f_k(x,u) + V_{k+1}(F_k(x,u))\} \qquad (3)$$

$k = N-1, N-2, \ldots, 0$ Then there exists an optimal solution to (1) and

- (a) $J_k^*(x) = V_k(x)$ for all $k = 0, 1, \ldots, N,\ x \in X_n$
- (b) The optimal feedback control in each stage is the minimizing argument in (3)

Proof, see Section 8.1.2.

Short Course on Gradients

$f(x+\delta x) = f(x) + (\nabla f)^T \delta x +$ higher order terms

For $f(x) = c^T x$ we get $\nabla f = c$

For $f(x) = x^T M x$ where M symmetric

$f(x+\delta x) = (x+\delta x)^T M(x+\delta x) =$
$x^T Mx + x^T M\delta x + \delta x^T Mx + \delta x^T M\delta x = x^T Mx + 2(x^T M)\delta x +$
higher order term

Hence $\nabla f = 2Mx$

Infinite Time Horizon Optimization

$$\begin{array}{ll} J^*(x_0) = \text{min} & \sum_{k=0}^{\infty} \gamma^k f(x_k, u_k) \\ \text{subject to} & x_{k+1} = F(x_k, u_k) \\ & x_0 \text{ given} \\ & u_k \in U(x_k) \end{array} \tag{4}$$

with $0 < \gamma \leq 1$ discount factor.

Bellman Equation

Assume $0 \in U(0)$, $f(0,0) = 0$, $F(0,0) = 0$ and that f is strictly positive definite. If there exists a strictly positive definite and quadratically bounded V such that the Bellman equation

$$V(x) = \min_{u \in U(x)} \{f(x,u) + \gamma V(F(x,u))\}$$

holds, then

- (a) $J^*(x) = V(x)$
- (b) The minimizing argument in the Bellman equation is an optimal feedback for (4) that results in a globally convergent closed loop system if γ is sufficiently close to one.