EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER4

TIME: Tuesday, August 26, 2025, 8–12

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

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VISITS: 9

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APPROVED TOOLS: Collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page after the exam.

The exam can be inspected and checked out 2025-09-11, 12.30-13.00, 2A:573, B-building, entrance 25, A-corridor, to the right.

PRELIMINARY GRADING:	grade 3	15 points
	grade 4	23 points
	grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) Find the control signal $u(t)$ which satisfies the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^T \Big(x_2(t) + u^2(t)\Big)\, dt \\ \text{subject to} \quad & \dot{x}_1(t) = -x_1(t) + u(t), \\ & \dot{x}_2(t) = x_1(t) \\ & x_1(0) = a,\ x_2(0) = 0, \end{aligned}$$

for a fixed $T > 0$, using the PMP. (5p)

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \left(y^2(t) + 4\dot{y}^2(t)\right) dt,$$

satisfying $y(0) = 1$ and $y(1) = 0$. (5p)

2. (a) The system

$$\dot{x} = u$$

is controlled to minimize the criterion

$$\int_0^\infty (x^{2m} + u^2)\, dt$$

where m is a positive integer. Derive a control law $u = \mu(x)$ that minimizes the criterion. You may use the fact that such a control law is the minimizing argument in the equation

$$0 = \min_u \left\{ x^{2m} + u^2 + \frac{\partial V(x)}{\partial x} u \right\}$$

where the function $V(x)$ is such that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$.

(5p)

(b) Consider an analogous discrete time problem. Minimize

$$\sum_{t=0}^{N-1} \Big(x(t)^{2m} + u(t)^2\Big) + x(N)^{2m}$$

for the system

$$x(t+1) = x(t) + u(t)$$

Write down the dynamic programming recursion for solving the problem. Explain why it is difficult to solve the problem explicitly if $m > 1$. (5p)

3. Consider the optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{\infty} x_k^2 + u_k^2 \\ \text{subject to} & x_{k+1} = x_k + u_k \\ & x_0 \text{ given} \\ & u_k \in [-1, 1] \end{array}$$

(a) Compute an optimal feedback policy $u_k = \mu(x_k))$ for this problem when the control signal constraint is neglected. *Hint:* try the value function $V(x) = px^2$, where $p > 0$. (4p)

(b) Now, consider the case with constraints on the control signal. Compute an approximative solution by solving

$$\min_{-1 \leq u \leq 1} \left\{ x^2 + u^2 + V(x + u) \right\}$$

where $V(x)$ is defined as in the hint above. (3p)

(c) Prove that the closed loop system using the feedback of the previous subproblem is stable. (3p)

4. We consider a linear system in discrete time:

$$x_{k+1} = Ax_k + v_k$$
$$y_k = Cx_k + e_k$$

where x_k and y_k are n and p dimensional vectors, repectively, and $k = 0, \ldots, N-1$. The initial state x_0 is given. We assume that the sequences of vectors v_k and e_k are zero mean identically distributed and independet random vectors with symmetric and postive definite covariance matrices R_1 and R_2, respectively. The likelihood function $L(\bar{x}, \bar{v})$ satisfies

$$-\log L(\bar{x}, \bar{v}) = \frac{1}{2} \sum_{k=0}^{N} (y_k - Cx_k)^T R_2^{-1} (y_k - Cx_k) + \frac{1}{2} \sum_{k=0}^{N-1} v_k^T R_1^{-1} v_k + c$$

where c is some constant, and where $\bar{x} = (x_1, \ldots, x_N)$ and $\bar{v} = (v_0, \ldots, v_{N-1})$. We are interested in solving the maximum likelihood problem of maximizing $L(\bar{x}, \bar{v})$ with respect to $(\bar{x}, \bar{v})$ under the constraint of the dynamic recursion above and for given measurements $\bar{y} = (y_0, \ldots, y_N)$.

Show that the optimal $\bar{x}$ satisfies the following two-point boundary problem for some sequence of Lagrange multipliers λ_k:

$$x_{k+1} = Ax_k - R_1 \lambda_{k+1}, \quad k = 0, \ldots, N-1$$
$$\lambda_k = A^T \lambda_{k+1} - C^T R_2^{-1} (y_k - Cx_k), \quad k = 1, \ldots, N-1$$

with boundary conditions x_0 given and $\lambda_N = -C^T R_2^{-1} (y_N - Cx_N)$. (10 p)