EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER4, U10

TIME: Wednesday, January 14, 2026, 14–18

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

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VISITS: 15

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APPROVED TOOLS: Collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page when the exam has been corrected.

PRELIMINARY GRADING:

grade 3	15 points
grade 4	23 points
grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) Solve the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^T \left(x(t) + u^2(t) \right) \, dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0. \end{aligned}$$

for a fixed $T > 0$, using the PMP. (5p)

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \left(y^2(t) + \dot{y}^2(t) \right) dt,$$

satisfying $y(0) = 0$ and $y(1) = 1$. (5p)

2. Consider the problem

$$\begin{aligned} \underset{u(\cdot)}{\text{maximize}} \quad & \sum_{k=0}^{N-1} \beta^k \log\left(u_k\right) \\ \text{subject to} \quad & x_{k+1} = a x_k^{\alpha} - u_k \geq 0, \\ & x_0 \geq 0 \text{ given}, \end{aligned}$$

where $0 < \alpha, \beta < 1$ and $a > 0$ are some constants. This is commonly referred to as the *consumption problem* in the theory of economics. The variable u_k may be interpreted as the consumption for time period k and x_k the available capital, which is assumed positive, at time period k, respectively. Find the optimal control policy u_k, where $k = 1, 0$, for the problem when the horizon is $N = 2$ using the dynamic programming algorithm. (10p)

3. We consider a discrete-time hidden Markov model with state $x(k)$ and output $y(k)$ at time k. We define the probability

$$p(\bar{x}) = P\{x(0) = x_0, \ldots, x(N-1) = x_{N-1}, y(0) = y_0, \ldots, y(N-1) = y_{N-1}\}$$

where $\bar{x} = (x_0, \ldots, x_{N-1}) \in X^N$, $\bar{y} = (y_0, \ldots, y_{N-1}) \in Y^N$, and where X and Y are some sets to be defined in more detail later. We assume that the sequence of observations $\bar{y}$ is given, and we are interested in estimating the sate sequence $\bar{x}$ based on the observation of $\bar{y}$. We are going to do this by maximizing $p(\bar{x})$, which is known as maximum likelihood estimation. A key feature of a hidden Markov model is that it satisfies what is called the Markov property, i.e.

$$\begin{aligned} p(\bar{x}) = {} & P\{y(N-1) = y_{N-1} | x(N-1) = x_{N-1}\} \\ & \times P\{x(N-1) = x_{N-1} | x(N-2) = x_{N-2}\} \\ & \times P\{y(N-2) = y_{N-2} | x(N-2) = x_{N-2}\} \\ & \times P\{x(N-2) = x_{N-2} | x(N-3) = x_{N-3}\} \\ & \vdots \\ & \times P\{y(1) = y_1 | x(1) = x_1\} P\{x(1) = x_1 | x(0) = x_0\} \\ & \times P\{y(0) = y_0 | x(0) = x_0\} P\{x(0) = x_0\} \end{aligned}$$

(a) Let $V_0(x) = P\{y(0) = y_0 | x(0) = x\} P\{x(0) = x\}$ and define the recursion

$$V_k(x) = P\{y(k) = y_k | x(k) = x\} \max_{u \in X} P\{x(k) = x | x(k-1) = u\} V_{k-1}(u)$$

for $k = 1, \ldots, N-1$, and let $V_N = \max_{x \in X} V(N-1, x)$. Show that

$$\max_{\bar{x} \in X^N} p(\bar{x}) = V_N$$

and that the optimal $\bar{x}$ is such that x_{k-1} is the maximizing u in iteration k above. *Hint:* It is a good idea to consider $\log p(\bar{x})$. (5p)

(b) Consider a primitive clinic in a village. People in the village have the property that they are either healthy or have a fever. They can only tell if they have a fever by asking the doctor in the clinic. The doctor makes a diagnosis of fever by asking patients how they feel. Villagers only answer that they feel normal, dizzy or cold. This defines a hidden Markov model with $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2, \beta_3\}$, where α_1 =healthy, α_2 =fever, β_1 =normal, β_2 =cold, and β_3 =dizzy. Introduce the notation:

$$a_{ij} = P\{x(k+1) = \alpha_j | x(k) = \alpha_i\}$$
$$b_{ij} = P\{y(k) = \beta_j | x(k) = \alpha_i\}$$

Also let the matrices A and B be defined such that element (i, j) of the matrix is equal to a_{ij} and b_{ij}, respectively. Consider the case when

$$A = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}; \quad B = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

and assume that $P\{x(0) = \alpha_1\} = 0.6$ and $P\{x(0) = \alpha_2\} = 0.4$. The doctor has for a patient observed the first day normal, the second day cold, and the third day dizzy. What is the most likely value of the condition for the patient, i.e. the most likely value of $\bar{x}$? Use the recursion in (a). (5p)

4. Consider the dynamical system

$$\frac{d^2y(t)}{dt^2} = u(t) + k$$

where $|u(t)| \leq 1$ is the control signal and where $0 < k < 1$ is a known constant. You should solve the optimal control problem

$$\underset{|u(t)|\leq 1}{\text{minimize}} \int_0^T |u(t)| dt$$

subject to the above dynamics with known initial values $y(0)$ and $\dot{y}(0)$ and such that $y(T) = 0$. The final time T should also be optimized.

(a) Use PMP to show that the optimal control signal is of the form

$$u(t) = \begin{cases} -1, & 0 \leq t \leq t_1 \\ 0, & t > t_1 \end{cases}$$

for some t_1. (4p)

(b) Show that the following feedback policy is optimal

$$u(y, \dot{y}) = \begin{cases} 0, & y \leq 0 \text{ or } y \leq \frac{\dot{y}^2}{2k} \text{ and } \dot{y} \leq 0 \\ -1, & \text{otherwise} \end{cases}$$

(4p)

(c) Consider the case where it is additionally required that $\dot{y}(T) = 0$. What form does the optimal control signal now have? (2p)