EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER3

TIME: Tuesday, March 17, 2026, 8–12

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 3

EXAMINER: Anders Hansson, 070-3004401

VISITS: 9

COURSE ADMINISTRATOR: Ninna Stensgård 013-282225, ninna.stensgard@liu.se

APPROVED TOOLS: Collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page when the exam has been corrected.

PRELIMINARY GRADING:

grade 3	15 points
grade 4	23 points
grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. (a) Find the control signal $u(t)$ expressed as a function of the states $x(t)$ which satisfies the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & x(T) + \int_0^T \frac{u^2(t)}{x(t)}\, dt \\ \text{subject to} \quad & \dot{x}(t) = -u(t), \end{aligned}$$

for a fixed T, using the PMP. (5p)

Hint: The adjoint equation is a separable ODE.

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \left(e^t y + \frac{\dot{y}^2}{t} \right) dt,$$

satisfying $y(0) = 1$ and $y(1) = 0$. (5p)

2. Assume that we have a vessel whose maximum weight capacity is z and whose cargo is to consist of different quantities of N different items. Let v_k denote the value of the kth type of item, and let w_k denote the weight of the kth type of item.

(a) Let x_k be the used weight capacity of the vessel after the first $k-1$ items have been loaded and let the control u_k be the quantity of item k to be loaded on the vessel. Formulate the dynamic equation

$$x_{k+1} = F(k, x_k, u_k),$$

describing the process. (3p)

(b) Determine the constraint set $U(k, x_k)$ on the control signal u_k. (3p)

(c) Formulate a DP recursion that solves the problem of finding the most valuable cargo satisfying the maximal weight capacity. Observe that you do *not* need to solve the problem. (4p)

3. Consider the scalar discrete time system

$$x_{k+1} = x_k + u_k, \quad x_0 = 1$$

with the optimization criterion

$$J = |x_N| + \sum_{k=0}^{N-1} |u_k|$$

(a) Consider the control law

$$u_k = -\alpha x_k, \quad 0 \leq \alpha \leq 1$$

Show that the cost is independent of the choice of α. What is the cost? (Hint: the sum of a geometric series is $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$) (5p)

(b) Use e.g. dynamic programming to show that the optimal cost satisfies

$$J_k^*(x_k) = |x_k|$$

for all time points $0 \leq k \leq N$. What can be concluded about optimality/suboptimality of the control law in (a)? (5p)

4. We consider a well stirred bio-reactor, where x is the concentration of bacteria, z is the concentration of contaminent, and where u is the concentration of input nutrient. The following differential equation model how the variables are related

$$\begin{aligned} \dot{x} &= \frac{Gu}{L+u}x - Dx^2 \\ \dot{z} &= -Kzx \end{aligned}$$

where D, G, K and L are postive constants. It is assumed that the initial values for x and z are positive. The objective is to make the $z(T)$ small at the same time as $\int_0^T u(t)dt$ is small, where $u(t) \in [0, M]$ for some positive constant M.

(a) Show that

$$\int_0^T Kx(t)dt = -\ln\left(\frac{z(T)}{z(0)}\right)$$

(2p)

(b) In view of (a) the objective is to maximize

$$\int_0^T (Kx - u)dt$$

under the above constraints. Let λ be the adjoint variable for this problem and define $z = x\lambda$. Show that the adjoint eqeuation implies that

$$\dot{z} = (zD - K)x, \quad z(T) = 0.$$

Moreover, show that that $0 < z(t) < K/D$ for all $t \in [0, T)$ and that $\dot{z}(t) < 0$ for all $t \in [0, T]$. Hint: $\dot{z} = (zD - K)x$ is separable. (3p)

(c) Show that the optimal control is

$$u^\star = \min\left(\max\left(0, \sqrt{GLz} - L\right), M\right)$$

and that the optimal solution will switch at most once from M to $\sqrt{GLz} - L$ and then to zero. (5p)