EXAM IN OPTIMAL CONTROL (TSRT08)

ROOM: TER1

TIME: Tuesday, August 25, 2026, 8–12

COURSE: TSRT08 OPTIMAL CONTROL

CODE: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover page): 3

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APPROVED TOOLS: Collections of formulas and tables printed by a publishing house, calculator. No other books are allowed.

SOLUTIONS: Linked from the course home page when the exam has been corrected.

PRELIMINARY GRADING:

grade 3	15 points
grade 4	23 points
grade 5	30 points

All solutions should be well motivated. Writing should be neat and clean.

Good Luck!

1. We are interested in computing optimal transportation routes in a circular city. The cost for transportation per unit length is given by a function $g(r)$ that only depends on the radial distance r to the city center. This means that the total cost for transportation from a point P_1 to a point P_2 is given by

$$\int_{P_1}^{P_2} g(r)ds$$

where s represents the arc length along the path of integration. In polar coordinates (θ, r) the total cost reads

$$\int_{P_1}^{P_2} g(r)\sqrt{1+(r\dot{\theta})^2}dr$$

where $\theta = \theta(r)$, and $\dot{\theta} = d\theta/dr$.

   (a) Formulate the problem of computing an optimal path as an optimal control problem (2p)

   (b) For the case of $g(r) = \alpha/r$ for some positive α show that any optimal path satisfies the equation $\theta = a\log r + b$ for some constants a and b. (5p)

   (c) Show that if the initial point and the final point are at the same distance from the origin, then the optimal path is a circle segment. You may use the claim in (b). (3p)

2. Consider the optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{\infty} x_k^2 + u_k^2 \\ \text{subject to} & x_{k+1} = x_k + u_k \\ & x_0 \text{ given} \\ & u_k \in [-1, 1] \end{array}$$

   (a) Compute an optimal feedback policy $u_k = \mu(x_k))$ for this problem when the control signal constraint is neglected. *Hint:* try the value function $J(x) = px^2$, where $p > 0$. (4p)

   (b) Now, consider the case with constraints on the control signal. Compute an approximative solution by solving

$$\min_{-1\le u\le 1} \left\{x^2 + u^2 + J(x+u)\right\}$$

   where $J(x)$ is defined as in the hint above. (3p)

   (c) Prove that the closed loop system using the feedback of the previous subproblem is stable. (3p)

3. Consider the system

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(t)$$

with the boundary conditions

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) Compute a control sequence that minimizes

$$u(0)^2 + u(1)^2 + u(2)^2$$

(6p)

(b) Compute instead a control sequence that minimizes

$$|u(2)|$$

(4p)

4. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \quad |u| \leq 1 \end{aligned}$$

with the minimization criterion

$$\int_0^4 \frac{u^2}{2} dt$$

and the boundary conditions

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(4) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Compute a function $u(t)$ that satisfies the Pontryagin Maximum Principle. (10p)