TSRT08: Optimal Control
Solutions

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1. (a) The Hamiltonian is given by

$$H(t,x,u,\lambda) = x_2 + u^2 + \lambda_1(-x_1 + u) + \lambda_2 x_1.$$

Pointwise minimization is obtained via

$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = 2u + \lambda_1 \quad \Rightarrow \quad u^* = -\frac{1}{2}\lambda_1,$$

since H is strictly convex in u. The adjoint equations are given by

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x_1}(t, x(t), \mu^*(t, x(t), \lambda(t)), \lambda(t)) = \lambda_1(t) - \lambda_2(t),$$
$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial x_2}(t, x(t), \mu^*(t, x(t), \lambda(t)), \lambda(t)) = -1$$

with boundary conditions

$$\lambda(T) = \frac{\partial \phi}{\partial x}(T, x(T)) \quad \Longleftrightarrow \quad \lambda_1(T) = \lambda_2(T) = 0$$

Thus, we get $\lambda_2(t) = T - t$ and

$$\dot{\lambda}_1(t) = \lambda_1(t) + t - T, \quad \lambda_1(T) = 0,$$

which has the solution

$$\lambda_1(t) = -(t - T) - 1 + e^{t-T}$$

and the optimal control is

$$u^*(t) = -\frac{1}{2}\lambda_1(t) = \frac{1}{2}\big(1 + t - T - e^{t-T}\big).$$

(b) Introducing $x = y$ and $u = \dot{y}$, it holds that $\dot{x} = u$ and the Hamiltonian is given by

$$H(t,x,u,\lambda) = x^2 + 4u^2 + \lambda u.$$

The following equations must hold

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(t,x,u,\lambda) = -2x,$$
$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = 8u + \lambda.$$

The latter yields $\dot{\lambda} = -8\dot{u} = -8\ddot{y}$, which substituted into the first equation gives

$$-8\ddot{y} = -2x = -2y \quad \Leftrightarrow \quad \ddot{y} - \frac{1}{4}y = 0,$$

which has the solution

$$y(t) = c_1 e^{t/2} + c_2 e^{-t/2},$$

for some constants c_1 and c_2. The boundary constraints $y(0) = 1$ and $y(1) = 0$ yields

$$\begin{pmatrix} 1 & 1 \\ e^{1/2} & e^{-1/2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which has the solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{e-1} \begin{pmatrix} -1 \\ e \end{pmatrix}.$$

Thus,

$$y(t) = \frac{1}{e-1}(-e^{t/2} + e^{1-t/2}),$$

is the sought after extremal.

2. (a) The problem gives the following Hamilton-Jacobi-equation

$$0 = \min_u (V_x u + u^2 + x^{2m}). \tag{1}$$

Minimizing with respect to u gives $u = -V_x/2$ and if you plug that into (1) one gets

$$0 = -\frac{V_x^2}{4} + x^{2m} \Rightarrow V_x = \pm 2|x|^m.$$

Since V shall be a value function we require $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$. This means that the sign of V_x shall be

$$V_x = \begin{cases} 2|x|^m & x > 0, \\ -2|x|^m & x < 0, \end{cases}$$

which gives $V(x) = 2|x|^{m+1}/(m+1)$. The optimal feedback is then

$$u = \begin{cases} -|x|^m & x > 0, \\ |x|^m & x < 0. \end{cases}$$

(b) The dynamic programming recursion is given by

$$\begin{aligned} V_N(x) &= \phi(x) \\ V_n(x) &= \min_u \{f(n, x, u) + V_{n+1}(F(n, x, u))\}, \end{aligned}$$

which is our case will be

$$\begin{aligned} V_N(x) &= x^{2m} \\ V_n(x) &= \min_u \{x^{2m} + u^2 + V_{n+1}(x + u)\}, \end{aligned}$$

For the case $m = 1$ this is a LQ problem (with finite time horizon). This can be solved with a cost-to-go function which is quadratic in x, this means with the ansatz $V_n(x) = \alpha_n x^2$. However, with $m > 1$, the polynomial order of $V_n(x)$ will inevitably increase as we proceed backward in time which makes it difficult to find an analytical solution.

3. (a) Neglecting the constraints on the control signal yields the Bellman equation

$$V(x) = \min_u \{f(x, u) + V(F(x, u))\},$$

which in our case with $V(x) = px^2$ can be written as

$$px^2 = \min_u \{\underbrace{x^2 + u^2 + p(x + u)^2}_{\triangleq h(x,u)}\}.$$

Differentiating h with respect to u and setting the derivative to zero yields

$$0 = \frac{\partial h}{\partial u}(x,u) = 2u + 2p(x+u) \quad \Longrightarrow \quad u^* = -\frac{p}{1+p}x,$$

which is a minimum since

$$\frac{\partial^2 h}{\partial u^2}(x,u^*) = 2 + 2p > 0.$$

Now, the Bellman equation reads

$$px^2 = x^2 + \left(-\frac{p}{1+p}x\right)^2 + p\left(x - \frac{p}{1+p}x\right)^2 = \frac{1+2p}{1+p}x^2,$$

which implies that $p^2 - p - 1 = 0$ and therefore

$$p = \frac{1+\sqrt{5}}{2}.$$

Finally, the optimal controller is given by

$$u^* = -\frac{\sqrt{5}-1}{2}x$$

(b) If $u \in (-1,1)$, then the solution is the same as in a). Thus,

$$u = \frac{1-\sqrt{5}}{2}x,$$

as long as x satisfies $-1 \leq (\sqrt{5}-1)x/2 \leq 1$. Otherwise, the solution lies on the boundary. Hence,

$$u = \begin{cases} 1, & x \leq -\frac{2}{\sqrt{5}-1} \\ \frac{1-\sqrt{5}}{2}x, & -\frac{2}{\sqrt{5}-1} \leq x \leq \frac{2}{\sqrt{5}-1} \\ -1, & x \geq \frac{2}{\sqrt{5}-1} \end{cases}.$$

(c) This problem may be divided into three subproblems:

(i) If $x_k \leq -2/(\sqrt{5}-1)$, then $u_k = 1$ and

$$x_{k+1} = x_k + u_k = x_k + 1 > x_k,$$

which shows that the state sequence is strictly increasing. Since

$$\text{width}\left[-\frac{2}{\sqrt{5}-1}, \frac{2}{\sqrt{5}-1}\right] > 1,$$

we know that $-2/(\sqrt{5}-1) \leq x_{k+n} \leq 2/(\sqrt{5}-1)$ after a finite number of steps n.

(ii) In a similar fashion, if $x_k \geq 2/(\sqrt{5}-1)$, then $u_k = -1$ and the state sequence is strictly decreasing and will reach the region of no saturation in a finite number of steps.

(iii) What remains to be shown is that the nominal controller

$$u = -\frac{\sqrt{5}-1}{2}x,$$

is stabilizing. It can be seen that for this case

$$x_{k+1} = x_k - \frac{\sqrt{5}-1}{2}x_k = \frac{3-\sqrt{5}}{2}x_k$$

which goes to zero as k goes to infinity since $|\frac{3-\sqrt{5}}{2}| < 1$. Moreover, it is clear that once the state reached a value where the nominal controller is activated, the nominal controller will not make the state leave this interval.

4. By interpreting v_k as control input, we can specify a discrete-time optimal control problem on the standard form

$$\begin{aligned} \text{minimize} \quad & \phi(x_N) + \sum_{k=0}^{N-1} f(k, x_k, v_k) \\ \text{subject to} \quad & x_{k+1} = F(k, x_k, v_k) \\ & x_0 \quad \text{given} \end{aligned}$$

where

$$\begin{aligned} \phi(x) &= \frac{1}{2}(y_N - Cx)^T R_2^{-1}(y_N - Cx) + c, \\ f(k, x, v) &= \frac{1}{2}(y_k - Cx)^T R_2^{-1}(y_k - Cx) + \frac{1}{2} v^T R_1^{-1} v, \\ F(k, x, v) &= Ax + v. \end{aligned}$$

We will use discrete-time PMP to solve the optimal control problem above. We do that by first defining the Hamiltonian, which is given by

$$\begin{aligned} H(k, x, v, \lambda) &= f(k, x, v) + \lambda^T F(k, x, v) \\ &= \frac{1}{2}(y_k - Cx)^T R_2^{-1}(y_k - Cx) + \frac{1}{2} v^T R_1^{-1} v + \lambda^T (Ax + v) \end{aligned}$$

Since R_1 is positive definite, also R_1^{-1} is positive definite. The quadratic form $v^T R_1^{-1} v$, and consequently also the Hamiltonian H, is therefore strictly convex in v. Pointwise minimization with respect to v then yields

$$\begin{aligned} 0 &= \frac{\partial H}{\partial v}(k, x, v, \lambda) = R_1^{-1} v + \lambda \Rightarrow \\ v &= -R_1 \lambda \end{aligned}$$

and the candidate optimal control is given by $v_k^* = -R_1 \lambda_{k+1}$.

The two-point boundary value problem is now given by

$$\begin{aligned} x_{k+1} &= \frac{\partial H}{\partial \lambda}(k, x_k, v_k^*, \lambda_{k+1}) = Ax_k + v_k^* = Ax_k - R_1 \lambda_{k+1}, \quad k = 0, \ldots, N-1 \\ \lambda_k &= -\frac{\partial H}{\partial x}(k, x_k, v_k^*, \lambda_{k+1}) = -C^T R_2^{-1}(y_k - Cx_k) + A^T \lambda_{k+1}, \quad k = 1, \ldots, N-1 \end{aligned}$$

with the boundary condition x_0 given and

$$\lambda_N = \frac{\partial \phi}{\partial x}(x_N) = -C^T R_2^{-1}(y_N - Cx_N).$$