TSRT08: Optimal Control
Solutions

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1. (a) The Hamiltonian is given by

$$H(t,x,u,\lambda) = \frac{u^2}{x} + \lambda(-u).$$

Pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = \frac{2u}{x} - \lambda \quad \Rightarrow \quad u^* = \frac{1}{2}\lambda x.$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t,x,u,\lambda) = \frac{u^2}{x^2} = \frac{1}{4}\lambda^2,$$

which is a separable ODE in λ. We have

$$\frac{d\lambda}{dt} = \frac{1}{4}\lambda^2 \quad \Leftrightarrow \quad \frac{1}{\lambda^2}d\lambda = \frac{1}{4}dt \quad \Leftrightarrow \quad -\frac{1}{\lambda} = \frac{1}{4}t + c \quad \Leftrightarrow \quad \lambda = -\frac{4}{t+4c}$$

for some constant c. Since $\lambda(T) = \frac{\partial \phi}{\partial x} = 1$, we have $c = -1 - T/4$ and thus $\lambda = -4/(t-T-4)$, which yields the optimal control signal

$$u^* = \frac{1}{2}\lambda x = -\frac{2}{t-T-4}x$$

(b) Introducing $x = y$ and $u = \dot{y}$, it holds that $\dot{x} = u$ and the Hamiltonian is given by

$$H(t,x,u,\lambda) = e^t x + u^2 t^{-1} + \lambda u.$$

The following equations must hold

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(t,x,u,\lambda) = -e^t, \tag{1a}$$

$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = 2ut^{-1} + \lambda. \tag{1b}$$

Now, (1a) yields $\lambda = -e^t + c_1$ and this inserted into (1b) yields

$$0 = 2ut^{-1} + \lambda = 2\dot{y}t^{-1} + c_1 - e^t \quad \Leftrightarrow \quad y = \frac{1}{2}\int t(e^t - c_1)\,dt = \frac{1}{2}(t-1)e^t - \frac{c_1}{4}t^2 + c_2.$$

Finally, $y(0) = 1$, $y(1) = 0$ gives the constants $c_1 = 6$ and $c_2 = 3/2$. Hence

$$y(t) = \frac{1}{2}\big((t-1)e^t - 3t^2 + 3\big),$$

is the sought after extrema.

2. (a) With $x_1 = 0$, the dynamic equation may be written as

$$x_{k+1} = x_k + w_k u_k.$$

(b) Since for each item k, we must have $x_k \leq z$ for all k and especially

$$x_{k+1} = x_k + w_k u_k \leq z \quad \Longleftrightarrow \quad u_k \leq (z - x_k)/w_k.$$

Furthermore, it is only possible to ship integer quantities which implies that $u \in \{0, 1, 2, \ldots\}$.

(c) The reward function is given by $f_0(k, x_k, u_k) = v_k u_k$, which yields the DP algorithm

$$V_{N+1}(x) = 0$$
$$V_n(x) = \max_{\substack{0 \leq u_n \leq (z-x_n)/w_n \\ u_n \in \{0,1,2,\ldots\}}} \Big\{ v_n u_n + V_{n+1}(x_n + w_n u_n) \Big\}, \quad n = N, N-1, \ldots, 1.$$

3. (a) With the proposed control law we have the recursion

$$x_{k+1} = (1 - \alpha) x_k, \quad x_0 = 1$$

which has the solution

$$x_k = (1 - \alpha)^k$$

This gives the cost

$$\begin{aligned} J &= |x_N| + \sum_{k=0}^{N-1} |u_k| \\ &= |x_N| + \sum_{k=0}^{N-1} |-\alpha x_k| \\ &= (1-\alpha)^N + \alpha \sum_{k=0}^{N-1} (1-\alpha)^k \end{aligned}$$

By using the formula for computing the sum of a geometric series we get

$$\sum_{k=0}^{N-1} (1-\alpha)^k = \frac{1 - (1-\alpha)^N}{1 - (1-\alpha)} = 1 - (1-\alpha)^N$$

which gives us cost

$$J = (1-\alpha)^N + 1 - (1-\alpha)^N = 1$$

This cost does not depend on α.

(b) The problem can be brought into the standard form by identifying $\Phi(x) = x$, $f(x, u) = |u|$ and $F(x, u) = x + u$. According to the dynamic programming algorithm we have

$$V_N(x) = \Phi(x) = |x|$$

Furthermore, assume that $V_{k+1}^*(x) = |x|$. This gives according to the DP-algorithm

$$\begin{aligned} V_k(x) &= \min_u \big\{ f(x, u) + V_{k+1}\big(F(x, u)\big) \big\} \\ &= \min_u \{ |u| + |x + u| \} \end{aligned}$$

By plotting $|u|$ and $|x + u|$ respectively, we realize that the sum $|u| + |x + u|$ has its minimum value $|x|$ when $-x \leq u \leq 0$ (see Figure 1). We then get

$$V_k(x) = \min_u \{ |u| + |x + u| \} = |x|$$

By induction we have then proven that $V_k(x) = |x|$ for all time points $0 \leq k \leq N$.
Furthermore, the optimal control law is

$$u_k^* = \arg\min_u \{ |u| + |x + u| \} = \tilde{\mu}(x)$$

where $\tilde{\mu}(x) \in [-x, 0]$. Since $-\alpha x \in [-x, 0]$ for $0 \geq \alpha \geq 1$, the proposed control law in 4a will be an optimal control.

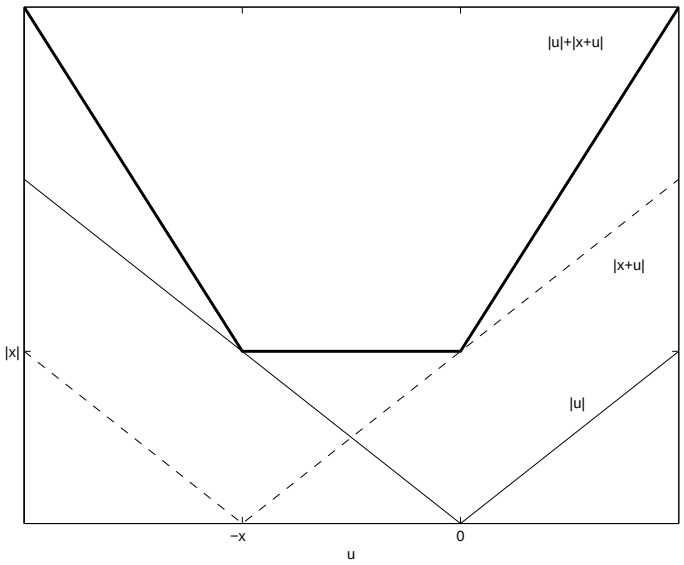


Figure 1: The quantities $|u|$, $|x+u|$ and $|u|+|x+u|$ as a function of u

4. (a) From $\dot{z} = -Kzx$ we have that $Kx = -\frac{\dot{z}}{z}$, which yields

$$\int_0^T Kx(t)dt = -\int_0^T \frac{\dot{z}(t)}{z(t)}dt = -\int_{z(0)}^{z(T)} \frac{1}{z}dz = -\left[\ln(z(T)) - \ln(z(0))\right] = -\ln\left(\frac{z(T)}{z(0)}\right)$$

(b) The Hamiltonian is given as

$$H(x,\lambda,u) = (Kx-u) + \lambda\dot{x} = (Kx-u) + \frac{Gu}{L+u}x\lambda - Dx^2\lambda,$$

which gives the adjoint equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -K - \frac{Gu}{L+u}\lambda + 2Dx\lambda. \tag{2}$$

With $z \triangleq x\lambda$ we get, using the product rule,

$$\begin{aligned}\dot{z} &= x\dot{\lambda} + \dot{x}\lambda \\ &= \left(-Kx - \frac{Gu}{L+u}x\lambda + 2Dx^2\lambda\right) + \left(\frac{Gu}{L+u}x\lambda - Dx^2\lambda\right) \\ &= (Dx\lambda - K)x \\ &= (Dz-K)x.\end{aligned} \tag{3}$$

Moreover, since there is no terminal constraint and $x(T)$ is free, the transversality condition is $\lambda(T) = 0$, which implies from the definition of z that $z(T) = 0$.
To show that $0 < z(t) < K/D$ we use the separability of (3), which gives

$$\begin{aligned}\int_{z(t)}^{z(T)} \frac{1}{Dz-K}dz &= \int_t^T x(s)ds \Leftrightarrow \\ \ln\left(\frac{-K}{Dz(t)-K}\right) &= D\int_t^T x(s)ds \Leftrightarrow \\ Dz(t) - K &= -Ke^{-D\int_t^T x(s)ds} \Leftrightarrow \\ z(t) &= \frac{K}{D}\left(1 - e^{-D\int_t^T x(s)ds}\right).\end{aligned} \tag{4}$$

Since $x(t) > 0 \, \forall t$ we have the inequality $0 < e^{-D\int_t^T x(s)ds} < 1$ for $t < T$, which in (4) gives the desired inequality $0 < z(t) < \frac{K}{D}$ for all $t \in [0, T)$. This inequality combined with (3) and $x(t) > 0$ implies that $\dot{z}(t) < 0$ for all $t \in [0, T)$.

(c) Recall that the Hamiltonian is $H(x, \lambda, u) = Kx - u + \frac{Gu}{L+u}x\lambda - Dx^2\lambda$. We have that

$$\frac{\partial H}{\partial u} = -1 + \frac{GL}{(L+u)^2}z \tag{5}$$

and that

$$\frac{\partial^2 H}{\partial u^2} = -2\frac{GL}{(L+u)^3}z < 0, \tag{6}$$

which follows from $z > 0$, $u \in [0, M]$ and $G, L > 0$. Hence, u such that $\frac{\partial H}{\partial u} = 0$ is a local maximum. Setting the LHS of (5) to zero yields

$$u = \pm\sqrt{GLz} - L, \tag{7}$$

where we can omit the negative square-root term since $u \in [0, M]$. We therefore end up with the optimal control

$$u^* = \begin{cases} 0, & \text{if } \sqrt{GLz} - L < 0 \\ \sqrt{GLz} - L, & \text{if } 0 \leq \sqrt{GLz} - L \leq M \\ M, & \text{if } \sqrt{GLz} - L > M, \end{cases} \tag{8}$$

which can be written compactly as

$$\min\left(\max\left(0, \sqrt{GLz} - L\right), M\right). \tag{9}$$

From (b) we know that $\dot{z} < 0$, which means that $\sqrt{GLz} - L$ strictly decreases over time. Hence, the possible switches in u^* need to be in the order $M \to \sqrt{GLz} - L \to 0$.