TSRT08: Optimal Control
Solutions

20260825

1. (a) By introducing the control $u = \dot{\theta}$, we can specify the optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \int_{r_1}^{r_2} g(r)\sqrt{1+(r\cdot u(r))^2}dr \\ \text{subject to} \quad & \dot{\theta} = u \\ & \theta(r_1) = \theta_1 \\ & \theta(r_2) = \theta_2 \end{aligned}$$

with variable u and where r_1, θ_1 and r_2, θ_2 are the polar coordinates of the points P_1 and P_2, respectively.

(b) With $g(r) = \alpha/r$, the Hamiltonian is given by

$$H(r,\theta,u,\lambda) = \frac{\alpha}{r}\sqrt{1+(r\cdot u)^2} + \lambda \cdot u$$

Further we have that

$$\begin{aligned} \frac{\partial H}{\partial u}(r,\theta,u,\lambda) &= \alpha\frac{r\cdot u}{\sqrt{1+(r\cdot u)^2}} + \lambda \\ \frac{\partial^2 H}{\partial u^2}(r,\theta,u,\lambda) &= \alpha\frac{r}{(1+(r\cdot u)^2)^{3/2}} \end{aligned}$$

Since $\alpha > 0$ and $r > 0$ we have that $\frac{\partial^2 H}{\partial u^2} > 0 \ \forall u$. Hence, $H(r,\theta,u,\lambda)$ is strictly convex in u. Therefor, pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(r,\theta^*(r),u^*(r),\lambda) = \alpha\frac{r\cdot u^*(r)}{\sqrt{1+(r\cdot u^*(r))^2}} + \lambda(r)$$

The adjoint equation is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial \theta}(r,\theta,u,\lambda) = 0$$

without final constraint on $\lambda(r_2)$ since we do have a final constraint on $\theta(r_2)$. This equation has the solution

$$\lambda(r) = c$$

for some constant c and the optimal control is

$$\alpha\frac{r\cdot u^*(r)}{\sqrt{1+(r\cdot u^*(r))^2}} = -c$$

This requires $r \cdot u^*(r)$ to be constant, which can be written as

$$ru^*(r) = a \quad \Rightarrow \quad u^*(r) = \frac{a}{r}$$

for some constant a. This gives the optimal path

$$\dot{\theta} = \frac{a}{r} \quad \Rightarrow \quad \theta = a \log r + b$$

which we were supposed to show.

(c) Reformulate the optimal path as a function of theta

$$r(\theta) = e^{\frac{\theta - b}{a}} = Be^{A\theta}$$

where $A = 1/a$ and $B = e^{-b/a}$. We now require that the initial and final point shall have the same radius r_0, such that $r(\theta_1) = r_0$ and $r(\theta_2) = r_0$. This gives

$$r_0 = Be^{A\theta_1}, \quad r_0 = Be^{A\theta_2} \quad \Rightarrow \quad A = 0, \quad B = r_0$$

which gives

$$r(\theta) = r_0, \quad \text{for} \quad \theta_1 \leq \theta \leq \theta_2$$

This corresponds to a path with constant radius, i.e. a circle segment.

2. (a) Neglecting the constraints on the control signal yields the Bellman equation

$$V(x) = \min_u \{f(x,u) + V(F(x,u))\},$$

which in our case with $V(x) = px^2$ can be written as

$$px^2 = \min_u \{\underbrace{x^2 + u^2 + p(x+u)^2}_{\triangleq h(x,u)}\}.$$

Differentiating h with respect to u and setting the derivative to zero yields

$$0 = \frac{\partial h}{\partial u}(x,u) = 2u + 2p(x+u) \quad \Longrightarrow \quad u^* = -\frac{p}{1+p}x,$$

which is a minimum since

$$\frac{\partial^2 h}{\partial u^2}(x,u^*) = 2 + 2p > 0.$$

Now, the Bellman equation reads

$$px^2 = x^2 + \left(-\frac{p}{1+p}x\right)^2 + p\left(x - \frac{p}{1+p}x\right)^2 = \frac{1+2p}{1+p}x^2,$$

which implies that $p^2 - p - 1 = 0$ and therefore

$$p = \frac{1+\sqrt{5}}{2}.$$

Finally, the optimal controller is given by

$$u^* = -\frac{\sqrt{5}-1}{2}x$$

(b) If $u \in (-1,1)$, then the solution is the same as in a). Thus,

$$u = \frac{1-\sqrt{5}}{2}x,$$

as long as x satisfies $-1 \leq (\sqrt{5}-1)x/2 \leq 1$. Otherwise, the solution lies on the boundary. Hence,

$$u = \begin{cases} 1, & x \leq -\frac{2}{\sqrt{5}-1} \\ \frac{1-\sqrt{5}}{2}x, & -\frac{2}{\sqrt{5}-1} \leq x \leq \frac{2}{\sqrt{5}-1} \\ -1, & x \geq \frac{2}{\sqrt{5}-1} \end{cases}.$$

(c) This problem may be divided into three subproblems:

(i) If $x_k \leq -2/(\sqrt{5}-1)$, then $u_k = 1$ and

$$x_{k+1} = x_k + u_k = x_k + 1 > x_k,$$

which shows that the state sequence is strictly increasing. Since

$$\text{width}\left[-\frac{2}{\sqrt{5}-1}, \frac{2}{\sqrt{5}-1}\right] > 1,$$

we know that $-2/(\sqrt{5}-1) \leq x_{k+n} \leq 2/(\sqrt{5}-1)$ after a finite number of steps n.

(ii) In a similar fashion, if $x_k \geq 2/(\sqrt{5}-1)$, then $u_k = -1$ and the state sequence is strictly decreasing and will reach the region of no saturation in a finite number of steps.

(iii) What remains to be shown is that the nominal controller

$$u = -\frac{\sqrt{5}-1}{2}x,$$

is stabilizing. It can be seen that for this case

$$x_{k+1} = x_k - \frac{\sqrt{5}-1}{2}x_k = \frac{3-\sqrt{5}}{2}x_k$$

which goes to zero as k goes to infinity since $|\frac{3-\sqrt{5}}{2}| < 1$. Moreover, it is clear that once the state reached a value where the nominal controller is activated, the nominal controller will not make the state leave this interval.

3. (a) The problem to be solved is

$$\begin{aligned} \text{minimize} \quad & \sum_{k=0}^{2} u^2(k) \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) \\ & x(0) = x_i \\ & x(3) = x_f \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \quad x_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We consider this as a constrained optimization problem with the parameter vector

$$u = \begin{bmatrix} u(0) & u(1) & u(2) \end{bmatrix}^T$$

and with the constraints

$$\begin{aligned} x_f &= A\Big(A\big[Ax_i + Bu(0)\big] + Bu(1)\Big) + Bu(2) \\ &= A^3x_i + A^2Bu(0) + ABu(1) + Bu(2) \end{aligned}$$

which gives

$$\begin{aligned} [A^2B,\, AB,\, B]\begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} &= x_f - A^3x_i \quad \Longrightarrow \\ \underbrace{\begin{bmatrix} 5/2 & 3/2 & 1/2 \\ 1 & 1 & 1 \end{bmatrix}}_{=:\tilde{A}} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}}_{=u} &= \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{=:\tilde{b}} \end{aligned} \tag{1}$$

Thus, we have the following quadratic program

$$\begin{aligned} \text{minimize} \quad & u^T u && \text{(2a)} \\ \text{subject to} \quad & \tilde{A}u = \tilde{b} && \text{(2b)} \end{aligned}$$

This QP can now be solved by reducing the optimization parameters. From (1) we have

$$\begin{cases} u(0) = -u(1) - u(2) \\ 5u(0) + 3u(1) + u(2) = -2 \end{cases} \Rightarrow 5(-u(1) - u(2)) + 3u(1) + u(2) = -2 \Rightarrow \begin{cases} u(1) = 1 - 2u(2) \\ u(0) = -1 + u(2) \end{cases} \tag{3}$$

which replaced into the cost function (2a) gives

$$u^T u = u^2(0) + u^2(1) + u^2(2) = (-1 + u(2))^2 + (1 - 2u(1))^2 + u^2(2) = 6u^2(2) - 6u(2) + 2$$

This is minimized by $u(2) = 1/2$. Using (3) this gives $u(0) = -1/2$ and $u(1) = 0$.

(b) We now consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & |u(2)| && \text{(4a)} \\ \text{subject to} \quad & \tilde{A}u = \tilde{b} && \text{(4b)} \end{aligned}$$

where $\tilde{A}$ and $\tilde{b}$ are defined as above. Since the optimization problem has three variables but only two constraints, the cost function (4a) can be brought to zero by requiring $u(2) = 0$, which obviously minimizes the cost function. Using this in (1) gives

$$\begin{bmatrix} 5/2 & 3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

4. The Hamiltonian is given by

$$H(t, x, u, \lambda) = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 u$$

Pointwise minimization yields

$$\tilde{\mu}(t, x, \lambda) = \underset{|u|<1}{\arg\min}\, H(t, x, u, \lambda) = \begin{cases} -1, & \lambda_2 > 1 \\ 1, & \lambda_2 < -1 \\ -\lambda_2, & -1 < \lambda_2 < 1 \end{cases} = -\text{sat}_{[-1,1]}\lambda_2$$

The adjoint equation is given by

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0, \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda_1(t) &= c_1 \\ \lambda_2(t) &= -c_1 t + c_2 \end{aligned}$$

Since we have full boundary constraints on both $x(0)$ and $x(4)$, there will be no boundary constraints on λ. We then get

$$u^*(t) = \text{sat}_{[-1,1]}(c_1 t - c_2) \tag{5}$$

for some unknown constants c_1 and c_2. To find a value on these constants we need to take the constraints $x(0) = [1\ 0]^T$ and $x(4) = [0\ 0]^T$ into consideration. Consequently, we have to find $x(t)$ when the control (5) is used. Now, assume that $|c_1 t - c_2| \leq 1$ for all t and use the control $u(t) = c_1 t - c_2$. For the x_2-state we then have

$$\begin{aligned} \dot{x}_2(t) &= u(t) = c_1 t - c_2 \\ x_2(0) &= 0 \end{aligned} \quad \Rightarrow \quad x_2(t) = \frac{c_1}{2} t^2 - c_2 t$$

which together with the boundary condition $x_2(4) = 0$ gives

$$x_2(4) = 8c_1 - 4c_2 = 0 \quad \Rightarrow \quad c_2 = 2c_1 \quad \Rightarrow \quad \begin{aligned} u(t) &= c_1(t-2) \\ x_2(t) &= c_1\left(\frac{t^2}{2} - 2t\right) \end{aligned}$$

This can now be used in the dynamics for $x_1(t)$

$$\begin{aligned} &\dot{x}_1(t) = x_2(t) = c_1\left(\frac{t^2}{2} - 2t\right) \\ &x_1(0) = 1 \end{aligned} \quad \Rightarrow \quad \dot{x}_1(t) = c_1\left(\frac{t^3}{6} - t^2\right) + 1$$

We now have

$$x_1(4) = c_1\left(\frac{64}{6} - 16\right) + 1 = -\frac{16}{3}c_1 + 1$$

which together with the boundary condition $x_1(4) = 0$ gives $c_1 = 3/16$. We thus get the control signal

$$u(t) = c_1 t - c_2 = c_1(t-2) = \frac{3}{16}(t-2)$$

Since this control signal obeys the constraint $|u(t)| \leq 1$ it will be equal to (5) and thus the one that obeys the PMP. Note that in this problem the constraint $|u| \leq 1$ will not affect the solution since it is never active.