Reglerteknik: Exercises

- **Exercises, Hints, Answers, Solutions**
- **Liten reglerteknisk ordlista (SV-ENG och ENG-SV)**
- **Introduktion till MATLAB och Control System Toolbox**

REGLERTEKNIK
AUTOMATIC CONTROL
LINKÖPING

This version: 2025-08-21

Exercises

This version: 2025-08-21

1 Mathematics

1.1 Determine the Laplace transform of the following signals.

a) A step-function, $u(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$, where A is a constant.

b) A ramp-function $u(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$, where A is a constant.

c) $u(t) = e^{-2t}$ for $t \geq 0$.

d) $u(t) = \cos 5t$ for $t \geq 0$.

Express the following in terms of $U(s)$, the Laplace transform of $u(t)$.

e) $\dot{u}(t)$

f) $\dot{u}(t)$, when $u(t) = 0$ for $t \leq 0$.

g) $\ddot{u}(t)$

h) $\ddot{u}(t)$, when $u(t) = \dot{u}(t) = 0$ for $t \leq 0$.

i) $u(t - T)$

Hint Answer Solution

1.2 Consider the differential equation

$$\dot{y}(t) + 2y(t) = u(t)$$

a) If $u(t)$ is constant and $y(t)$ converges to a constant value, then $\dot{y}(t) \approx 0$ when time goes to infinity. What value will $y(t)$ approach as $t \to \infty$ if $u(t) = 5$?

b) Determine the *transfer function* relating $U(s)$ and $Y(s)$ for the differential equation above. Assume $y(0) = 0$.

c) Solve the problem in (a) using the *final value theorem* and the result in (b).

Hint Answer Solution

1.3 What are the time functions corresponding to the Laplace transforms below? What values will the time functions approach as time goes to infinity?

a)

$$F(s) = \frac{1}{s^2 + s}$$

b)

$$F(s) = \frac{1}{s^2 - 1}$$

c)

$$F(s) = \frac{1}{(s + 1)^2}$$

Hint Answer Solution

1.4 Below, differential equations that describe dynamic systems are given together with system inputs and initial conditions. Use the Laplace transform to determine the solution $y(t)$.

a)

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = u(t)$$

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$\frac{\mathrm{d}y(0)}{\mathrm{d}t} = y(0) = 0$$

b)

$$\dot{y}(t) + y(t) = u(t)$$

$$u(t) = 1 + \sin t$$

$$y(0) = 0$$

Hint Answer Solution

1.5 The water level, $y(t)$, in a tank with an outlet hole is modelled by the differential equation

$$\dot{y}(t) + y(t) = z(t)$$

where $z(t)$ denotes the inflow. The inflow is a function of a *valve* position, which in turn is controlled by the electric control signal $u(t)$. The relation between control signal and resulting inflow is given by the differential equation

$$\ddot{z}(t) + \dot{z}(t) + z(t) = u(t)$$

What differential equation relates the water level $y(t)$ to the control signal $u(t)$? You can assume all initial conditions to be 0.

Hint Answer Solution

1.6 Verify that a quadratic $s^2 + as + b$ has roots with negative real part if and only if $a > 0$ and $b > 0$

Hint Answer Solution

1.7 Write the following complex numbers in polar form, that is, determine their absolute value and *argument* (angle). Draw the points in the complex plane and indicate the geometric meaning of the absolute value and *argument*.

a) $1 + \mathrm{i}$

b) $\frac{1+\mathrm{i}}{5\mathrm{i}(1+\sqrt{3}\mathrm{i})}$

Write the following complex numbers on rectangular form, and draw the points in the complex plane and indicate the geometric meaning of the absolute value and *argument*.

c) $2e^{\mathrm{i}\frac{\pi}{3}}$

d) $5e^{-\mathrm{i}\pi}$

Hint Answer Solution

1.8 A system has gain 100. What is the gain expressed in decibel (dB_{20})? What is the gain corresponding to 20 dB_{20}, -3 dB_{20}, 0 dB_{20}, -10 dB_{20}, and 10 dB_{20} respectively?

Hint Answer Solution

1.9 Verify that the following rule for inversion of 2×2 matrices holds.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Hint **Answer** **Solution**

2 Dynamic Systems

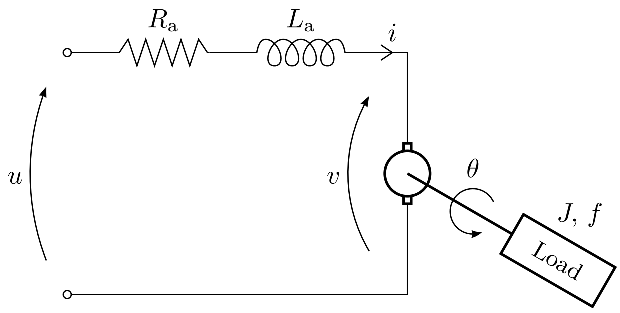


Figure 2.1a

2.1 A common component in a control system is a *DC-motor*, schematically depicted in Figure 2.1a. The motor is characterized by a number of physical relationships as will now be explained. The rotating axis is described by

$$J\ddot{\theta}(t) = -f\dot{\theta}(t) + M(t),$$

where $\theta(t)$ is the angle of rotation, $M(t)$ is the applied *torque*, J is the *moment of inertia* of the load and f is a friction coefficient.

Kirchhoff's *voltage* law states that

$$u(t) - R_a i(t) - L_a \frac{\mathrm{d}i(t)}{\mathrm{d}t} - v = 0$$

The interplay between rotor and stator in the motor is given by

$$M(t) = k_a i(t) \quad \text{and} \quad v(t) = k_v \dot{\theta}(t)$$

where $i(t)$ is the *current*, k_a a proportional constant characteristic for the motor, $v(t)$ is *voltage* generated by the rotating axis and k_v is a proportional constant. The input *voltage* $u(t)$ is the control signal and $\theta(t)$ is the output.

a) Use the equations above to write a differential equation that relates $u(t)$ and $\theta(t)$. The inductance L_a can be neglected and set to 0.

b) Determine the *transfer function* of the system from $u(t)$ to $\theta(t)$.

c) Study the behavior of the system by calculating $\theta(t)$ when $u(t)$ is a step.

Hint Answer Solution

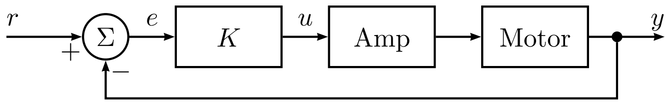


Figure 2.2a

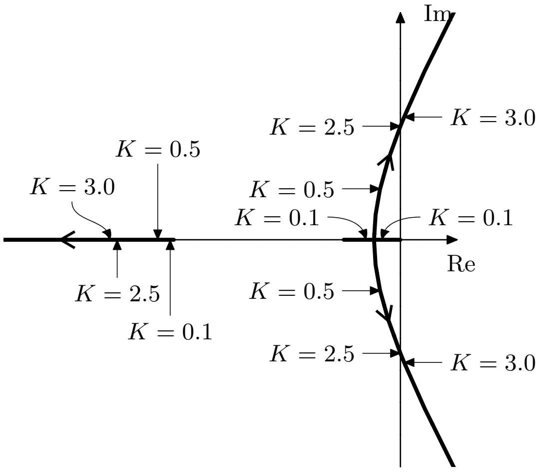


Figure 2.2b

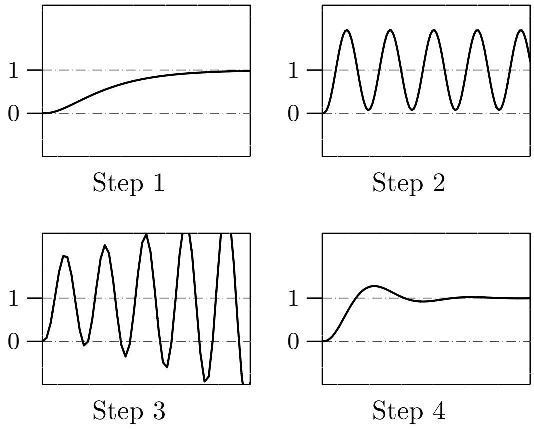


Figure 2.2c. All comparable axes have equal scaling.

2.2 A servo system for positioning of a tool in a tooling machine is depicted in Figure 2.2a. In Figure 2.2b, the poles (there are three of them) of the closed loop system are plotted as a function of K , and locations are indicated for $K = 0.1, K = 0.5, K = 2.5, K = 3.0$. Find (without calculations), for each of the *step responses* in Figure 2.2c, the corresponding value of K used when generating the *step response*.

Hint Answer Solution

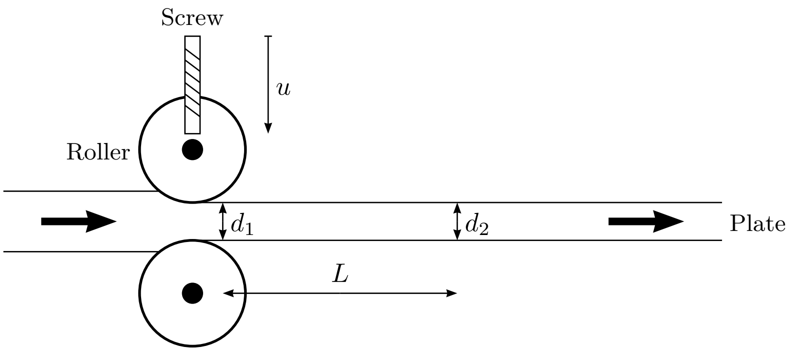


Figure 2.3a

2.3 Consider the simple model of the *rolling mill* depicted in Figure 2.3a. To obtain a simple model we describe the relationship between the position of the screw $u(t)$ and the thickness $d_1(t)$ of the *sheet* directly after the rollers as a first order *transfer function*

$$G(s) = \frac{\beta}{1 + sT}$$

To determine the constants β and T we register the effect of a sudden change in the position of the screw. The units used in the model are chosen such that a *unit step* on the screw position will make an appropriately sized input for identification purposes, and that is the input used in the experiment for which the resulting thickness profile $d_1(t)$ is shown in Figure 2.3b.

a) Find the *transfer function* from the position of the screw to the thickness d_1, i.e., identify the parameters β and T.

b) In production the thickness cannot be measured directly behind the rollers for practical reasons, and instead the thickness $d_2(t)$ is measured L length units after the rollers. Find the *transfer function* from the position of the screws to the thickness d_2. The sheet moves with speed V.

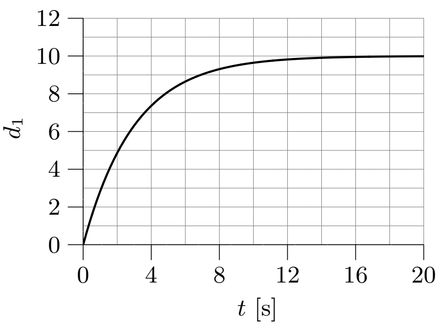


Figure 2.3b

Hint **Answer** **Solution**

2.4 The *step response* of the following system

$$G(s) = \frac{1}{s^2 + s + 1}$$

is shown as the dashed line in each part of Figure 2.4a.

a) The *step response* of the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

is shown as the solid line in the left of Figure 2.4a. Determine if $a > 1$ or $a < 1$.

b) The *step response* of the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

is shown as the solid line in the right of Figure 2.4a. Find b.

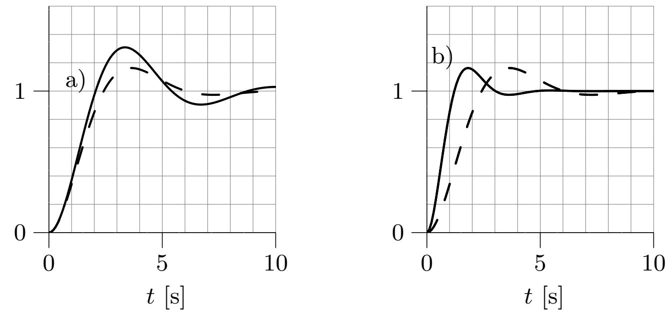


Figure 2.4a. Dashed: original system. Solid left: part a). Solid right: part b).

Hint Answer Solution

2.5 Pair the *step responses* and pole-zero diagrams in Figure 2.5a. Poles are marked by an `x` and zeros are marked by a `o`. The dashed line in the *step response* corresponds to 0.

Hint Answer Solution

2.6 The purpose of this exercise is to learn simple commands to analyze linear systems and understand the connection between the poles of a system and the resulting dynamical response.

When working in MATLAB, it is recommended to open a file and save everything in scripts (write `edit` in the command prompt). That way you can easily copy-paste code and quickly create all the models and plots.

Define the symbolic variable `s = tf( 's' );` and you can define the systems using this variable with standard operators +-*/. To simulate step-responses with a *unit step*, use the command `step`. The command can take multiple arguments to plot several step responses. After plotting, right click in the figure and select "Characteristics" if you want to see information about *rise time*, *settling time* etc. To compute poles of a system use `pole`. To analyze the computed poles, use `abs` for absolute value and `angle` for the *argument*. Remember that the angle is in radians. Another way to study poles is by the command `pzmap` applied to a system. Note that you can send several systems at the same time to both `step` and `pzmap`.

Consider the systems

$$G_{\mathrm{A}}(s) = \frac{5}{s^2 + 6s + 5} \qquad G_{\mathrm{B}}(s) = \frac{25}{s^2 + 10s + 25}$$

$$G_{\mathrm{C}}(s) = \frac{25}{s^2 + 5s + 25} \qquad G_{\mathrm{D}}(s) = \frac{100}{s^2 + 10s + 100}$$

$$G_{\mathrm{E}}(s) = \frac{25}{s^2 + 1s + 25} \qquad G_{\mathrm{F}}(s) = \frac{25}{s^2 + 4s + 25}$$

a) Plot the *step responses* of $G_A(s)$ and $G_B(s)$ (in the same figure). How do they differ? Compute the poles of the two systems. What do you see?

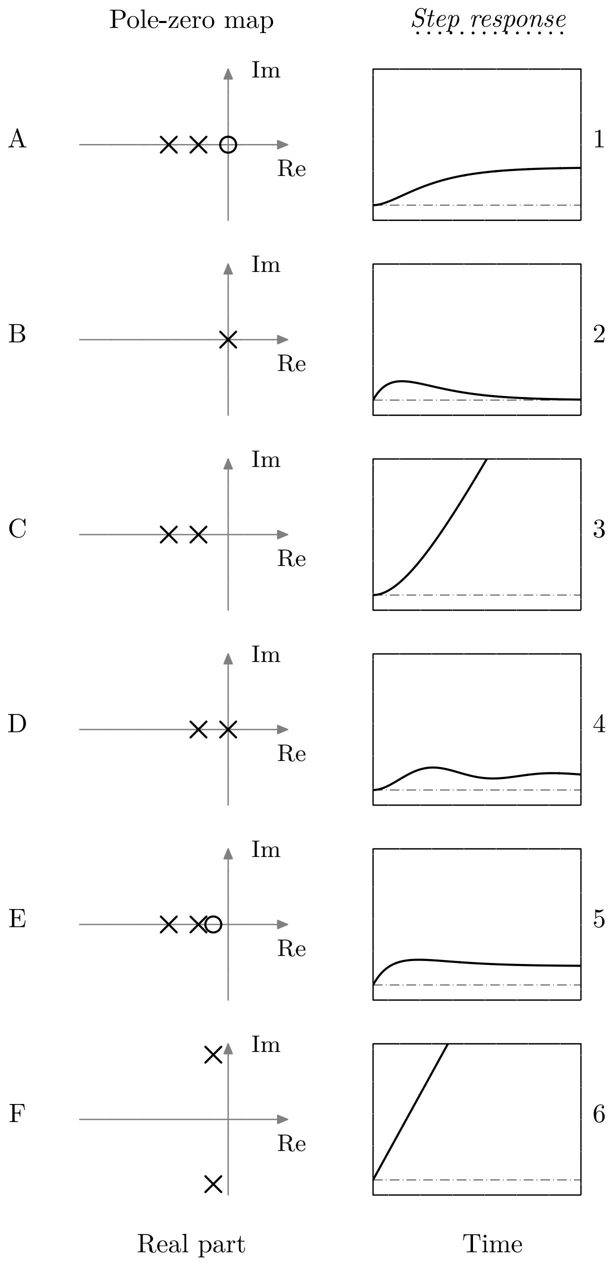


Figure 2.5a. All comparable diagrams have equal scaling. In the pole-zero maps, imaginary and real parts have equal scaling, $\times$ marks poles, and $\circ$ marks zeros.

b) Plot the step responses of $G_C(s)$ and $G_D(s)$. How do they differ? Compute the poles of the two systems and compute their absolute values and arguments (and plot them using `pzmap`). What do you see?

c) Plot the step responses of $G_E(s)$ and $G_F(s)$. How do they differ? Compute the poles of the two systems and compute their absolute values and arguments (and plot them using `pzmap`). What do you see?

d) Plot the step response of the product $G_A(s)G_E(s)$. What do you see and why?

Hint Answer Solution

2.7 Consider a system with the transfer function

$$G(s) = \frac{\alpha s + 1}{s^2 + 2s + 1}$$

Compute and plot the step response of the system for some different values of α in the range $-10 < \alpha < 10$. Where is the zero of the transfer function (root of numerator) located as a function of α, and how are the properties of the step response affected by the location of the zero?

Hint Answer Solution

2.8 Figure 2.8a shows the step response of a system $Y(s) = G(s)U(s)$. The input step has amplitude 1. Use the figure and determine

a) Steady state value.

b) Overshoot M in % of the final value.

c) Rise time T_r.

d) Settling time T_s.

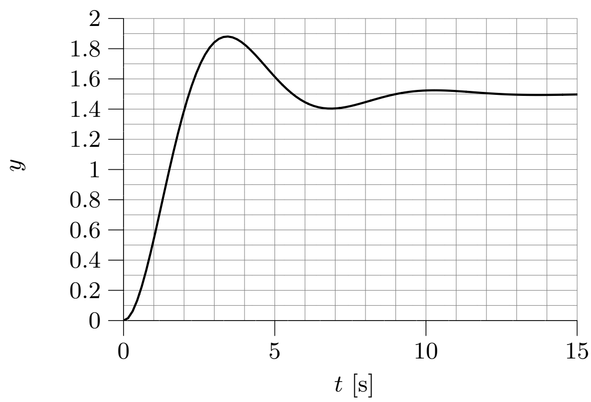


Figure 2.8a

Hint Answer Solution

2.9 Figure 2.9a shows the step responses of four different systems. Combine each step response with a transfer function from the alternatives below.

transfer function	Poles	Zeros	$\lvert G(0)\rvert$
$G_1(s) = \frac{100}{s^2+2s+100}$	$-1 \pm 10i$		1
$G_2(s) = \frac{1}{s+2}$	-2		1/2
$G_3(s) = \frac{10s^2+200s+2000}{(s+10)(s^2+10s+100)}$	$-10, -5 \pm 8.7i$	$-10 \pm 10i$	2
$G_4(s) = \frac{200}{(s^2+10s+100)(s+2)}$	$-2, -5 \pm 8.7i$		1
$G_5(s) = \frac{600}{(s^2+10s+100)(s+3)}$	$-3, -5 \pm 8.7i$		2
$G_6(s) = \frac{400}{(s^2-10s+100)(s+2)}$	$-2,\ 5 \pm 8.7i$		2

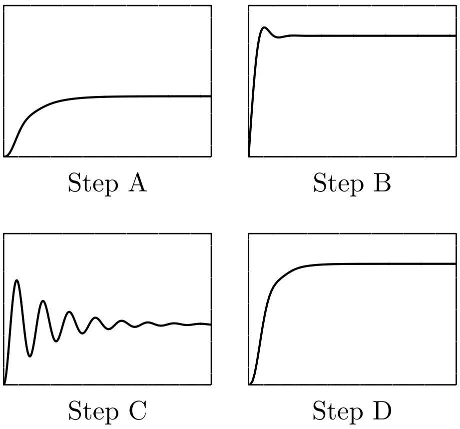


Figure 2.9a. All comparable axes have equal scaling.

Hint Answer Solution

3 Feedback Systems

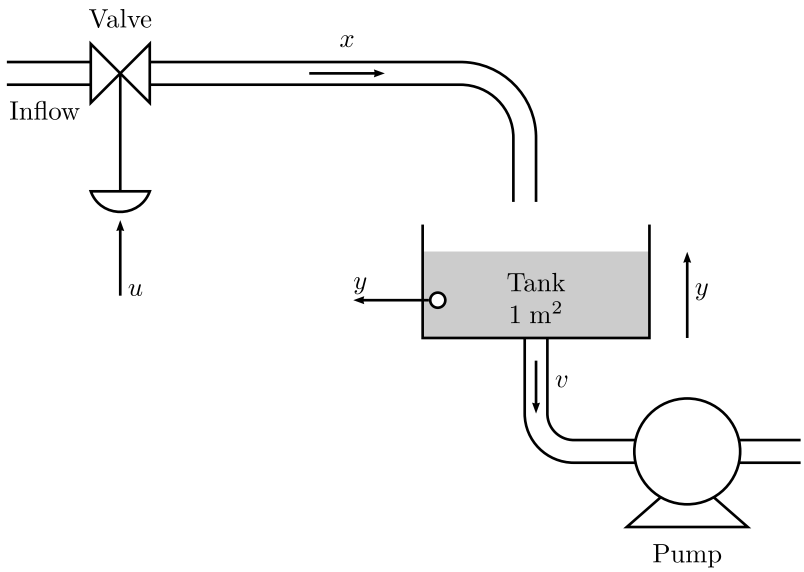


Figure 3.1a

3.1 Consider a part of a process industry, consisting of a valve and a tank. The system is described by Figure 3.1a and the following information:

- The valve is operated electrically, and the input voltage is denoted u. The resulting liquid flow is denoted x.
- The level in the tank is denoted y, and the flow out from the tank, v, is determined by a pump located further down the process. We have no control over this flow so $v(t)$ is a disturbance.
- The pipes are always filled with incompressible liquid, so there are no transport delays in the system.
- The cross section area of the tank is 1 m^2.

a) Use mass balance, that is, the fact that the change in volume per time unit is proportional to the difference between inflow and outflow, to determine a transfer function $G_\mathrm{t}(s)$ for the tank from the net inflow $x(t) - v(t)$ to the resulting level $y(t)$.

b) The transfer function of the valve from input voltage $u(t)$ to flow $x(t)$ is assumed to be

$$G_\mathrm{v}(s) = \frac{k_\mathrm{v}}{1 + Ts}$$

To find k_v and T a unit step change in u has been applied. The step response, that is, the resulting liquid flow, is shown in Figure 3.1b. Determine the constants k_v and T.

c) Draw a block diagram of the system, in which the valve and the tank are represented by one block each, and the signals u, x, y and v are indicated.

Hint Answer Solution

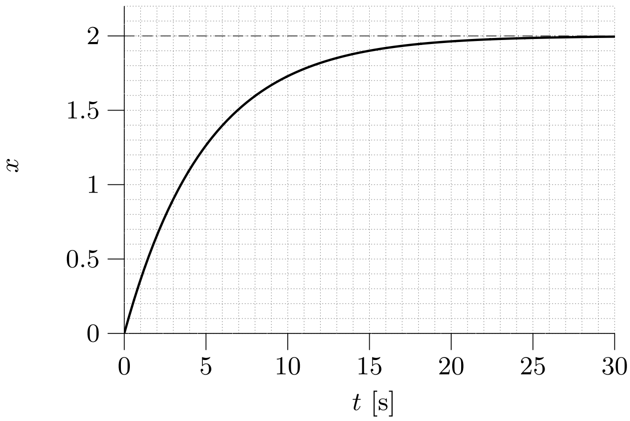


Figure 3.1b

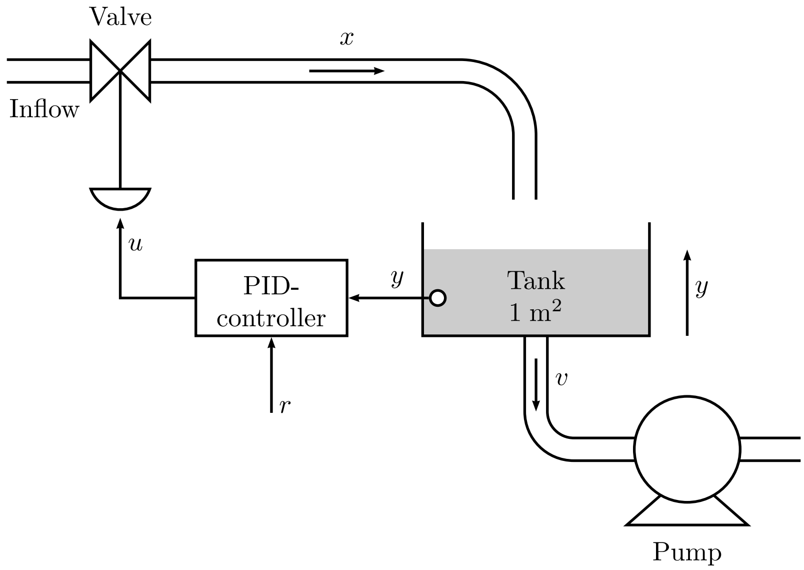


Figure 3.2a

3.2 Consider the system studied in Problem 3.1. The aim is now to control the tank level automatically, and therefore PID *feedback* control is introduced according to Figure 3.2a. This means that the *valve* input is determined according to the expression

$$u(t) = K_{\mathrm{P}} e(t) + K_{\mathrm{I}} \int_0^t e(\tau)\,\mathrm{d}\tau + K_{\mathrm{D}} \frac{\mathrm{d}e(t)}{\mathrm{d}t} \tag{3.1}$$

where $e(t)$ denotes the control error, that is,

$$e(t) = r(t) - y(t)$$

and $r(t)$ denotes the reference level, that is, desired tank level.

a) Verify, using Laplace transform, that the relationship in (3.1) can be expressed

$$U(s) = F(s)(R(s) - Y(s))$$

where

$$F(s) = K_{\mathrm{P}} + K_{\mathrm{I}} \frac{1}{s} + K_{\mathrm{D}} s$$

b) Let the PID controller $F(s)$ be represented by a block and draw a block diagram of the entire feedback control system including $G_V(s), G_t(s)$ and all signals of interest e, u, x, v and y.

c) Use the block diagram from above together with the transfer functions $G_{\mathrm{v}}(s)$, $G_{\mathrm{t}}(s)$, and $F(s)$, and derive the transfer functions of the closed loop system, that is, the transfer functions from the reference level $r(t)$ and flow $v(t)$ to the tank level $y(t)$.

Hint Answer Solution

3.3 Consider the level control system studied in Problems 3.1 and 3.2.

a) Assume that the tank level is controlled using proportional control, that is, $F(s) = K_{\mathrm{P}}$. Compute the poles of the closed loop system for $K_{\mathrm{P}} = 0.02$ and $K_{\mathrm{P}} = 1$ respectively. How does the choice of K_{P} affect the properties of the closed loop system?

b) Assume that the reference level is constant, $r(t) = 5$, and that the disturbance outflow is constant, $v(t) = 2$. Determine the steady state tank level when the proportional feedback $F(s) = K_{\mathrm{P}}$ is used. Is it possible for the output signal $y(t)$ to reach the desired level? What happens with the steady state level and other properties of the closed loop system if K_{P} is chosen very large?

c) Assume that a PI controller is used, that is,

$$u(t) = K_{\mathrm{P}} e(t) + K_{\mathrm{I}} \int e(\tau)\,\mathrm{d}\tau$$

What can be said about the steady state tank level in this case (you may assume the controller parameters have been selected so that the closed loop system is stable)? What is the possible benefit of introducing the integrating part in the feedback?

d) Finally, assume that a PD controller is used, that is,

$$u(t) = K_{\mathrm{P}} e(t) + K_{\mathrm{D}} \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Assume also that $K_{\mathrm{P}} = 1$ and calculate a value of K_{D} so the damping ratio of the closed loop poles will be greater than $1/\sqrt{2}$. This corresponds to placing the poles in the grey area in Figure 3.3a. What is the benefit of including the derivative part in the feedback?

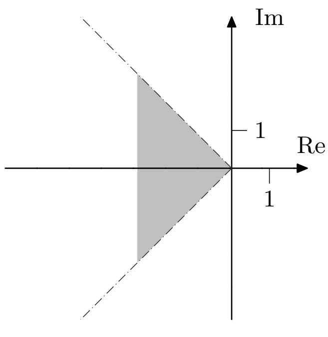


Figure 3.3a

Hint Answer Solution

3.4 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)} U(s).$$

a) Suppose $G(s)$ is controlled by a proportional controller with gain K_P, that is,

$$U(s) = K_P(R(s) - Y(s)).$$

Use MATLAB to compute the closed loop system, and to plot the *step response* of the closed loop system. Choose some values for K_P in the range 0.1 to 10. How are the properties of the *step response* affected by K_P? What happens with the *steady state error* when K_P increases? Is it possible to obtain a well damped closed loop system and small *steady state error* using proportional control? To form a closed loop system $\frac{FG}{1+FG}$, use the command `feedback(F*G,1)`. An alternative when you only want to plot *step responses* is to simply write `F*G/(1+F*G)` but the problem with this direct sapproach is that MATLAB often fails to reduce it to a minimal *transfer function* so some of the poles and zeros will be reported twice when computing these. Try both and look at the results.

b) Let us now introduce integration in the regulator and use

$$U(s) = (K_P + K_I\frac{1}{s})(R(s) - Y(s)).$$

Put $K_P = 1$ and try some values of K_I in the range $0 < K_I < 2$. How are the *step response* and the *steady state error* affected by the introduction of the integrating part and the value of K_I?

c) Finally we will introduce the differentiating part in the regulator and use

$$U(s) = (K_P + K_I\frac{1}{s} + \frac{K_D s}{sT+1})(R(s) - Y(s)).$$

Since true differentiation is difficult to implement, the derivative part is approximated by $\frac{K_D s}{1+sT}$. (This will low-pass filter the error signal before differentiation.) Put $K_P = 1, K_I = 1$ and $T = 0.1$ and try some values of K_D in the range $0 < K_D < 3$. How does the D-part affect the *step response* of the closed loop system?

Hint Answer Solution

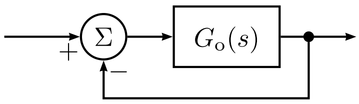


Figure 3.5a

3.5 Draw a *root locus* with respect to K for the system in Figure 3.5a, with $G_o(s)$ given below. For which values of K are the systems stable? What conclusions on the principal shape of the *step response* can be drawn from the *root locus*?

a) A Ferris wheel (Swedish: *Pariserhjul*):

$$G_o(s) = \frac{K(s+2)}{s(s+1)(s+3)}$$

b) A Mars rover:

$$G_o(s) = \frac{K}{s(s^2+2s+2)}$$

c) A magnetic floater:

$$G_o(s) = \frac{K(s+1)}{s(s-1)(s+6)}$$

Hint Answer Solution

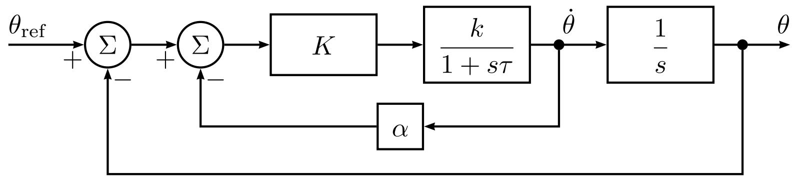


Figure 3.6a

3.6 Consider the servo system in Figure 3.6a with a *DC-motor*. Suppose that the angular velocity can be measured with a *tachometer* and let the control law be as in the block diagram. Let $\tau = 0.5$ and $k = 2$.

a) Draw the *root locus* with respect to K for the system without the *tachometer feedback* (that is, $\alpha = 0$).

b) Draw the *root locus* with respect to K for $\alpha = 1$.

c) Draw the *root locus* with respect to K for $\alpha = 1/3$.

d) Let $K = 1$ and draw the *root locus* with respect to α.

Discuss, using the results from a), b), c), and d), what is gained by using the *tachometer*.

Hint Answer Solution

3.7 We want to control the temperature of an unstable chemical reactor. The *transfer function* is

$$\frac{1}{(s+1)(s-1)(s+5)}$$

a) Use a proportional controller and draw a *root locus* with respect to the *amplification* K. Calculate which K in the compensator that stabilizes the system.

b) Use a PD controller. The control law is given by

$$u = K(e + T_{\text{D}}\frac{\text{d}e}{\text{d}t})$$

where e is the error. Let $T_{\text{D}} = 0.5$ and draw a *root locus* with respect to K. For which values of K does the controller stabilize the system?

Hint Answer Solution

3.8 Figure 3.8a shows the *root locus* for the characteristic equation of a P-controlled process G with respect to the gain K. In Figure 3.8b four *step responses* for the closed loop system with different values of K are shown. Match the plots in Figure 3.8b with the K-values below. Justify your answer.

$$K = 4 \quad K = 10 \quad K = 18 \quad K = 50$$

Hint Answer Solution

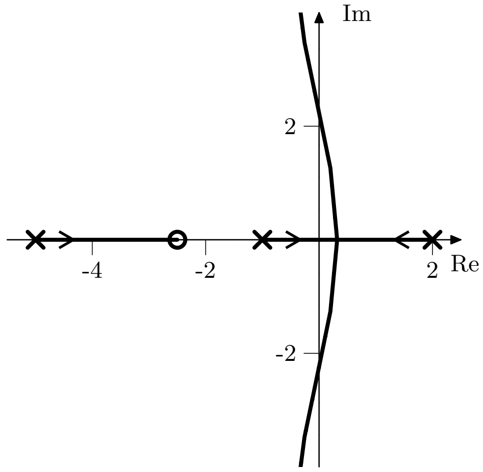


Figure 3.8a. Starting points are marked × and end points ○.

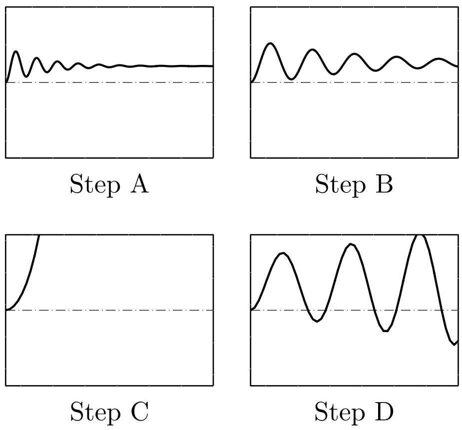


Figure 3.8b. All comparable axes have equal scaling.

3.9 Consider a system with the *transfer function*

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \cdots + b_{n-1}}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

that has all zeros strictly in the left half plane. Show that such a system always can be stabilized by

$$u(t) = -Ky(t)$$

if K is selected large enough.

Hint Answer Solution

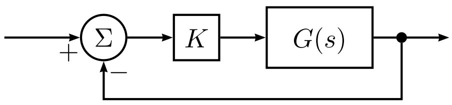


Figure 3.10a

3.10 A system $G(s)$ is controlled using *feedback* with a proportional controller according to Figure 3.10a.

a) For $K = 1$, the open loop system $KG(s)$ has the Nyquist diagram according to (i), (ii), (iii), or (iv) in Figure 3.10b. Is the closed loop system stable in each case? $G(s)$ has no poles in the right half plane.

b) If $K > 0$, for which values of K are the different closed loop systems stable?

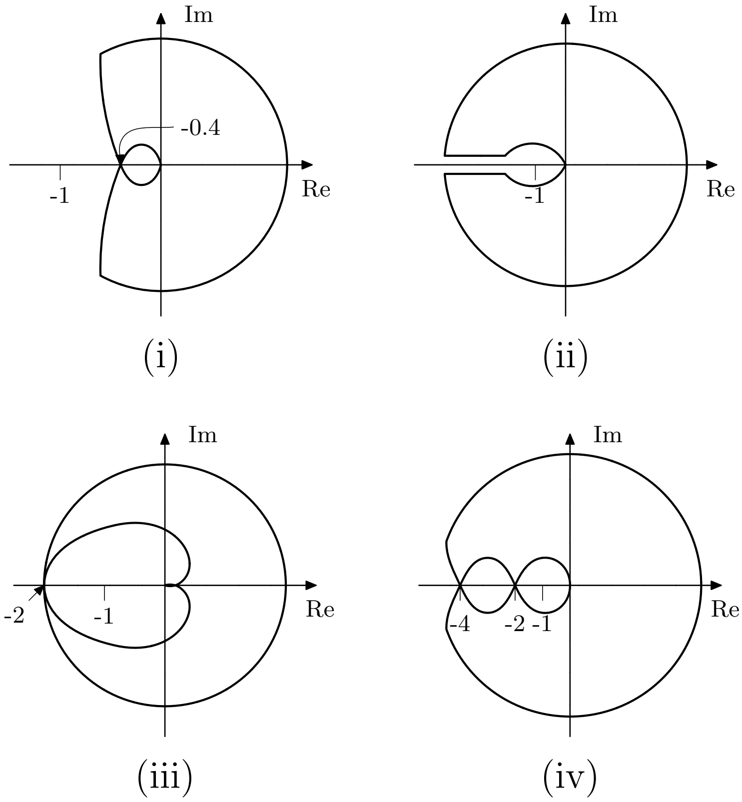


Figure 3.10b

Hint Answer Solution

3.11 a) Draw the Nyquist curve for an integrator $G(s) = 1/s$.

b) Draw the Nyquist curve for the double integrator $G(s) = 1/s^2$.

Hint Answer Solution

3.12 The system $G(s)$ is asymptotically stable and has the Nyquist curve in Figure 3.12a. It is controlled using *feedback* according to Figure 3.12b.

a) For what values of K ($K > 0$) is the closed loop system asymptotically stable?

b) Determine the *steady state error*, e, as a function of K if y_{ref} is a *unit step*.

c) Assume that G is controlled using an I controller according to Figure 3.12c. For what values of K is the closed loop system stable?

Hint Answer Solution

3.13 The equations for the P, PI, and PID controllers to be used in this problem are given in Problem 3.4.

In MATLAB, the command for drawing the root-loci of the characteristic equation $P(s)+KQ(s)) = 0$ is `rlocus(Q/P)`.

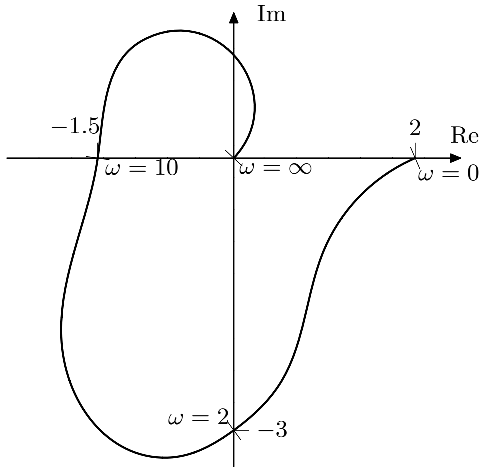


Figure 3.12a

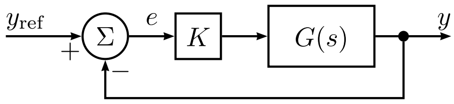


Figure 3.12b

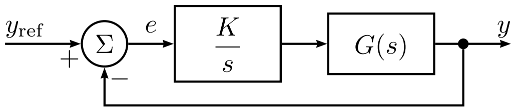


Figure 3.12c

a) Let the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2+s+1)(s+0.2)}U(s)$$

be controlled by a proportional controller with gain K_P. Use MATLAB to plot the *root locus* with respect to K_P of the characteristic equation of the closed loop system. For which values of $K_P > 0$ is the closed loop system asymptotically stable?

In Problem 3.4, we found that the *step response* was slow but well damped for small values of K_P, while it became faster but more oscillatory when K_P was increased. For large values of K_P the system became unstable. We also found that the *steady state error* was reduced when K_P was increased. Can these results be interpreted using the plot of the *root locus*?

b) Now assume that the system is controlled by a PI controller where $K_P = 1$. Plot the *root locus* of the characteristic equation, with respect to K_I, and determine for which values of $K_I > 0$ the closed loop system is asymptotically stable.

Problem 3.4 showed that an integrator eliminates the *steady state error*. A small value of K_I gives a large *settling time*, while a too large value gives an oscillatory, and perhaps unstable closed loop system. Give an interpretation of these results using the *root locus*.

c) Finally, let the system be controlled by a PID controller where $K_P = 1$, $K_I = 1$ and $T = 0.1$. Plot a *root locus* of the closed loop characteristic equation, with respect to $K_D > 0$, and relate the behavior of the *root locus* to the simulation result in Problem 3.4, that is, that the derivative part increases the damping of the closed loop system, but a too large K_D will give an oscillation with a higher frequency, and finally an unstable closed loop system.

Hint Answer Solution

3.14 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2+s+1)(s+0.2)}U(s).$$

a) Use MATLAB to plot the Nyquist curve of the open loop system when $G(s)$ is controlled by a proportional regulator. Try some different values of K_P and find for which K_P the closed loop system is asymptotically stable. Compare your results with those from Problem 3.13a.

b) Assume now that the system is controlled by a PI controller where $K_\mathrm{P} = 1$. Investigate how K_I affects the Nyquist curve and determine for which values of K_I the closed loop system is asymptotically stable. Do you get the same results as in Problem 3.13b?

c) Finally test a PID controller with $K_\mathrm{P} = 1$, $K_\mathrm{I} = 1$ and $T = 0.1$ (cf Problem 3.4). How is the Nyquist curve affected by the value of K_D?

Hint **Answer** **Solution**

3.15 a) Assume that the system

$$Y(s) = G(s)U(s) = \frac{0.4}{(s^2+s+1)(s+0.2)}U(s)$$

is controlled by a proportional controller where $K_\mathrm{P} = 1$. Use MATLAB to make a Bode plot of the open loop system and determine ω_c (*gain crossover frequency*), ω_p (*phase crossover frequency*), φ_m (*phase margin*) and A_m (*gain margin*) respectively. Compute the closed loop system and plot the *step response*.

b) Now let $K_\mathrm{P} = 2.5$. How does the change of K_P affect ω_c, ω_p, φ_m, and A_m ? Simulate the *step response* of the closed loop system and plot the result. How have the properties of the *step response* changed?

c) How much can K_P be increased before the closed loop system becomes unstable? How does this value relate to the value of A_m that was obtained for $K_\mathrm{P} = 1$? Compute and plot the *step response* of the closed loop system for this value of K_P. How does the closed loop system behave in this case?

Hint **Answer** **Solution**

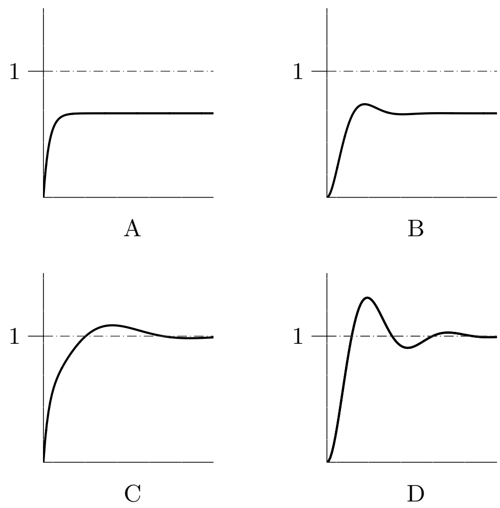


Figure 3.16a. Four step responses. All comparable axes have equal scaling.

Figure 3.16b

3.16 A system is controlled by a PID controller,

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)E(s)$$

In Figure 3.16a four step responses from a unit step for the parameter triples

$$\begin{array}{rlll} i) & K_P = 1 & K_I = 0 & K_D = 0 \\ ii) & K_P = 1 & K_I = 1 & K_D = 0 \\ iii) & K_P = 1 & K_I = 0 & K_D = 1 \\ iv) & K_P = 1 & K_I = 1 & K_D = 1 \end{array}$$

are shown. Match each one of the parameter triples to one of the step responses. Justify your answer!

Hint Answer Solution

Go back

3.17 Assume that a DC-motor of the type described in Problem 2.1 is controlled by a proportional controller, that is, $u(t) = K_P(\theta_{ref} - \theta)$.

a) Write down a block diagram for the control system. Compute the closed loop transfer function and determine how the poles of the closed loop system depend on the control gain K_P. Discuss what this means for the behavior of the system for different values of K_P.

b) Determine the transfer function from the reference signal to the error. Let the reference signal be a step and a ramp respectively and determine what the control error will be in steady state in these two cases.

c) Let the controller be a PI controller. What will the steady state error be in this case if the reference signal is a ramp?

Hint Answer Solution

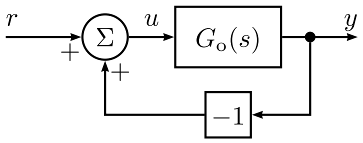


Figure 3.18a

3.18 Determine the *transfer function* for the *loop gain* and the closed loop system for the control system given by the block diagram in Figure 3.18a.

Hint Answer Solution

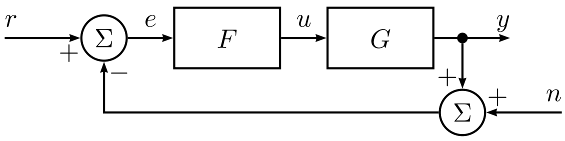


Figure 3.19a

3.19 Figure 3.19a shows a block diagram of a control system. Determine the *transfer function*

a) of the *loop gain*,
b) of the closed loop system from $R(s)$ to $Y(s)$,
c) from the measurement error $N(s)$ to the output $Y(s)$,
d) from the reference signal $R(s)$ to the error signal $E(s)$.

Hint Answer Solution

3.20 Consider again the control system in Figure 3.19a, with $n = 0$ and

$$G(s) = \frac{1}{(s+1)(s+3)}$$

a) Assume $F(s) = K$. Determine the steady state control error when $r(t)$ is a step with amplitude A.
b) Determine a regulator $F(s)$ such that the *steady state error* is zero when $r(t)$ is a step amplitude A.
c) Assume $F(s) = 1$. Determine the poles and zeros of the closed loop system.

Hint Answer Solution

3.21 The system

$$Y(s) = \frac{1}{(s/0.6+1)(s+1)}U(s)$$

is controlled using PID feedback

$$U(s) = (K_P + K_I\frac{1}{s} + K_D s)(R(s) - Y(s))$$

Figure 3.21a shows the *step responses* for the following four combinations of coefficient values. Combine the *step responses* and coefficients.

(1) $K_P = 4 \quad K_I = 0 \quad K_D = 0$
(2) $K_P = 4 \quad K_I = 3 \quad K_D = 0$
(3) $K_P = 4 \quad K_I = 1 \quad K_D = 0$
(4) $K_P = 4 \quad K_I = 0 \quad K_D = 4$

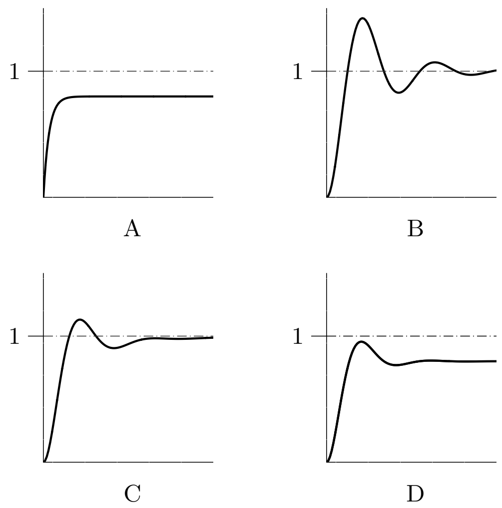


Figure 3.21a. Four step responses. All comparable axes have equal scaling.

Hint Answer Solution

4 Frequency Description

4.1 A mercury thermometer can be described with high accuracy as a first order linear time invariant dynamic system $\frac{a}{s+b}$. The input is the real temperature and the output is the thermometer reading. In order to decide the *transfer function* in a thermometer it is placed in liquid where the temperature is varied as a sinusoid around a constant level. The obtained result is shown in Figure 4.1a. Find the *transfer function* of the thermometer, considering the input and output as deviations around the constant levels.

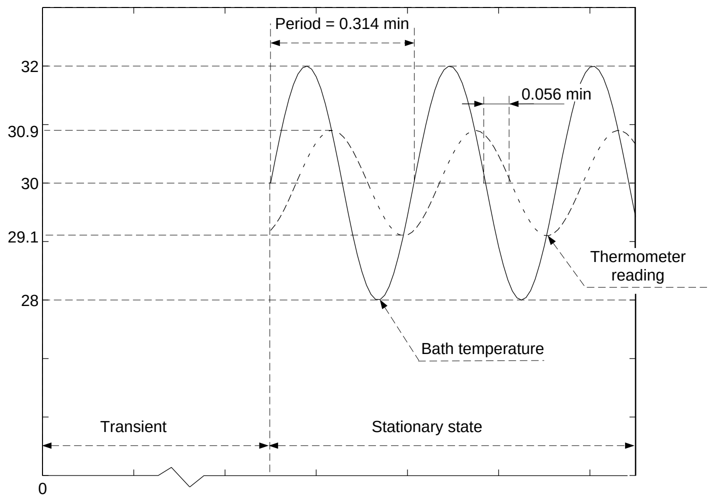


Figure 4.1a

Hint Answer Solution

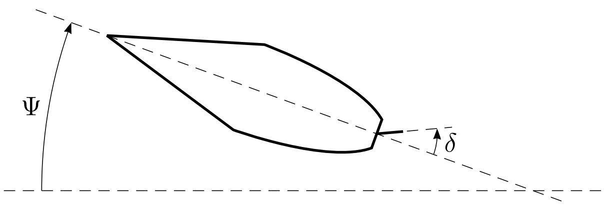


Figure 4.2a

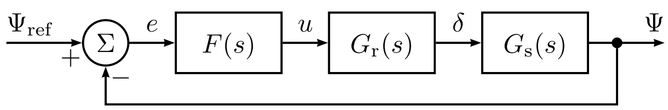


Figure 4.2b

4.2 We want to keep a ship on a given course, Ψ, with an automatic control system using the rudder angle δ. See Figure 4.2a. If ω denotes the angular velocity of the ship,

$$\omega = \dot{\Psi} \tag{4.1}$$

the following differential equation is valid for small values of ω and δ,

$$T_1\dot{\omega} = -\omega + K_1\delta \tag{4.2}$$

where $T_1 = 100$ and $K_1 = 0.1$. The desired course, Ψ_{ref}, and the measured course, Ψ, are fed in to the auto pilot, which gives the signal u to the rudder engine. Figure 4.2b shows a block diagram of the auto pilot. The auto pilot has the *transfer function*

$$F(s) = K\frac{1+\frac{s}{a}}{1+\frac{s}{b}}, \quad a = 0.02,\, b = 0.05$$

while G_r is given by

$$G_\text{r}(s) = \frac{1}{1+sT_2}, \quad T_2 = 10$$

and $G_\text{s}(s)$ is defined by (4.1) and (4.2).

a) Make a Bode plot for the *transfer function* $FG_\text{r}G_\text{s}$, for $K = 0.5$.

b) At the testing of the auto pilot we do the following experiment. The gain of the auto pilot K is increased until the control system oscillates with constant amplitude. At what value of K does this occur? What is the period time of the oscillation?

c) Ψ_{ref} is allowed to vary as a sinusoid

$$\Psi_{\text{ref}}(t) = A\sin\alpha t$$

where $A = 5°$ and $\alpha = 0.02$. When the movements of the ship have stabilized we have

$$\Psi(t) = B\sin(\beta t + \varphi)$$

What values do B, β, and φ have if $K = 0.5$?

Hint **Answer** **Solution**

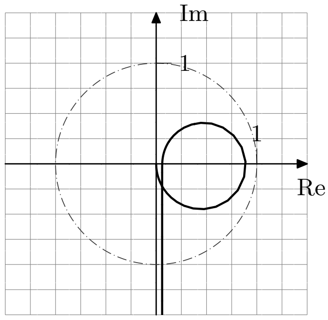


Figure 4.3a

4.3 a) In Figure 4.3a the Nyquist curve for a system is shown. Draw the Bode plot for the same system. The scale on the ω-axis is not important, as long as the amplitude and phase curve are in agreement.

b) Draw a diagram for the poles and zeros for the system. The relative placement is important, not the scale.

Hint Answer Solution

4.4 Figure 4.4a shows the *step responses* (when the input is a *unit step*) and Bode gain plots of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each step response, find the corresponding Bode gain plot (amplitude curve). Motivate your answer by pointing out a set of unique features for each system.

Hint Answer Solution

4.5 a) Assume we have the following closed-loop *transfer functions*.

$$G_\mathrm{A}(s) = \frac{5}{s^2+6s+5} \qquad G_\mathrm{B}(s) = \frac{25}{s^2+10s+25}$$
$$G_\mathrm{C}(s) = \frac{25}{s^2+5s+25} \qquad G_\mathrm{D}(s) = \frac{100}{s^2+10s+100}$$
$$G_\mathrm{E}(s) = \frac{25}{s^2+1s+25} \qquad G_\mathrm{F}(s) = \frac{25}{s^2+4s+25}$$

Study the amplitude curves of the Bode plots (using the commands `bode` or `bodemag`) for the systems and find the *static gain* and bandwidth of the systems. In cases when it is relevant find also the resonance frequency and resonance peak.

b) Describe qualitatively (without formulas) the relationships between T_r (*rise time*) and ω_B (bandwidth) and between M (*overshoot*) and M_p (resonance peak) respectively by performing step responses on the systems using `step`.

Hint Answer Solution

4.6 A system has the *transfer function*

$$G(s) = \frac{e^{-2s}}{s(s+1)}$$

What is the output (after transients) when the input is

$$u(t) = 2\sin(2t - 1/2)$$

Hint Answer Solution

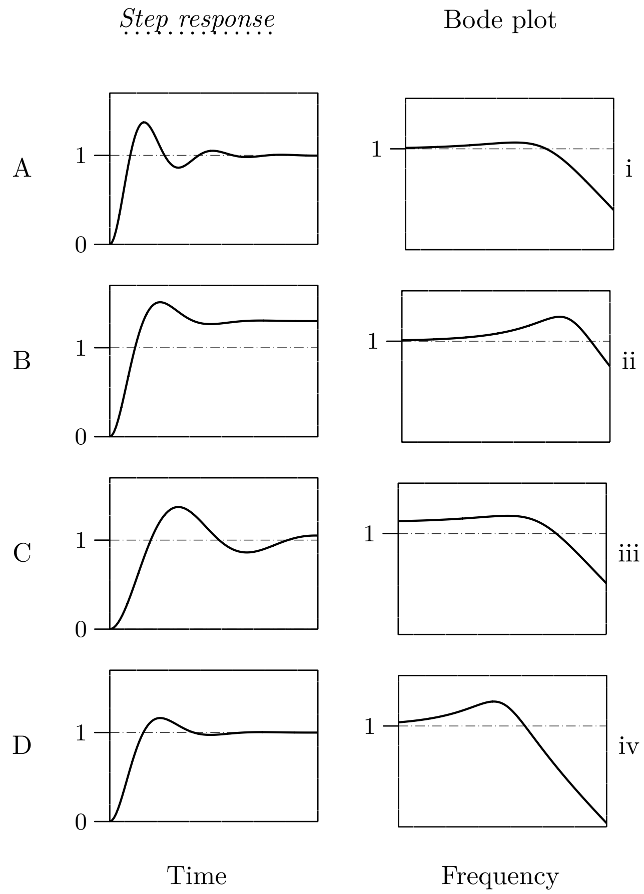


Figure 4.4a. All comparable diagrams have equal scaling.

4.7 For the systems below the input is chosen as $u(t) = \sin(2t)$. Determine the output signal $y(t)$ after transients have faded away, provided that it exists.

a) $Y(s) = \frac{1}{s+1}U(s)$

b) $Y(s) = \frac{1}{s-1}U(s)$

c) $Y(s) = \frac{1}{(s+1)(2s+1)}U(s)$

d) $Y(s) = \frac{e^{-0.5s}}{s+1}U(s)$

Hint Answer Solution

4.8 A system is described by $Y(s) = G(s)U(s)$. Figure 4.8a shows $u(t) = \sin(\omega t)$ and the corresponding output $y(t)$ (after all transients have faded away) for the frequencies $\omega = 1$, 5, 10, and 20 rad/s (from top to bottom).

a) Determine the gain ($|G(i\omega)|$) and phase ($\arg G(i\omega)$) for the system for each value of ω.

b) Determine the gain values in dB_{20} $(20 \log_{10}(|G(i\omega)|))$.

c) Sketch the Bode plot using the values determined above.

Hint Answer Solution

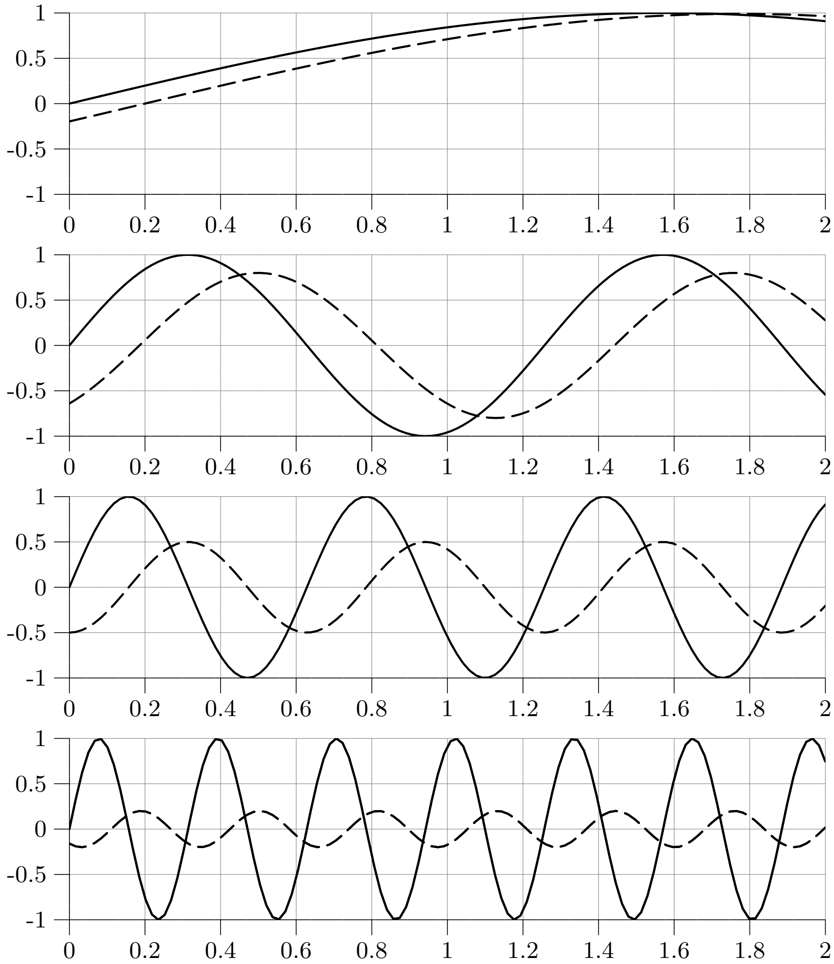


Figure 4.8a. $u(t) = \sin(\omega t)$ (solid) and $y(t)$ (dashed).

4.9 Combine the *transfer functions* below with the Bode plots in Figure 4.9a.

$$G_1(s) = \frac{1}{s+1}, \qquad G_2(s) = \frac{6(s+1)}{(s+2)(s+3)}$$

$$G_3(s) = \frac{1}{s}, \qquad G_4(s) = \frac{1}{s(s+1)}$$

$$G_5(s) = \frac{5}{s^2+2s+5} \text{ (poles: } -1 \pm \mathrm{i}2)$$

Hint Answer Solution

4.10 Figure 4.10a shows the Bode gain plots and *step responses* of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each Bode gain plot (amplitude curve), find the corresponding *step response*. Motivate your answer by pointing out a set of unique features for each system.

Hint Answer Solution

4.11 A system

$$Y(s) = G_o(s)U(s)$$

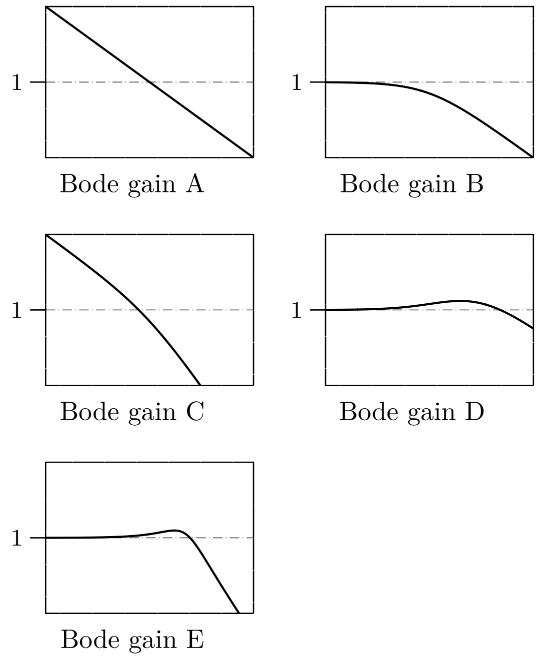


Figure 4.9a. All diagrams have equal scaling.

is used with a controller

$$U(s) = R(s) - Y(s).$$

a) In Figure 4.11a are Bode diagrams of the open loop systerm G_o for two different systems (top), the closed loop systems G_c (second row), step responses of the closed loop systems (third row) and the poles of the closed loop systems (bottom). Assign each closed loop system, closed loop step response and closed loop pole diagram to the correct open loop system.

b) Solve the same task for Figure 4.11b.

Hint Answer Solution

4.12 A system is described by

$$Y(s) = G(s)U(s)$$

where the transfer function from the input u to the output y is given by

$$G(s) = \frac{\frac{s}{\alpha} + 1}{s + 1} \quad \alpha > 0$$

a) Determine the pole and zero of the system.

b) Determine the static gain of the system for the values $\alpha = 2$ and $\alpha = 0.5$.

c) Determine $\mid G(i\omega) \mid$. What value does the function tend to when $\omega \to \infty$?

d) Make simple sketches of $\mid G(i\omega) \mid$ for the values $\alpha = 2$ and $\alpha = 0.5$. How does the location of the zero relative to the location of the pole affect the shape of $\mid G(i\omega) \mid$?

Hint Answer Solution

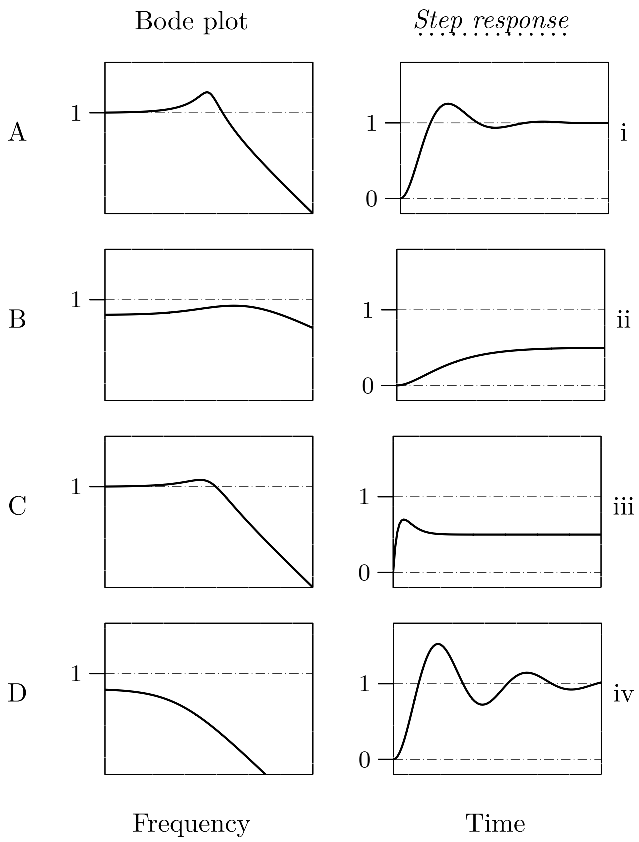
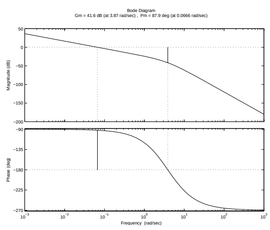
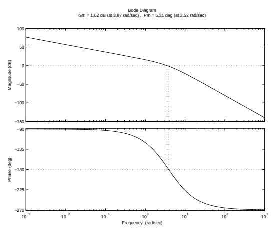


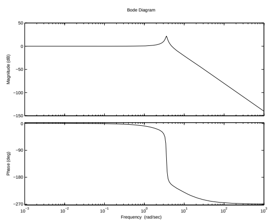
Figure 4.10a. All comparable diagrams have equal scaling.



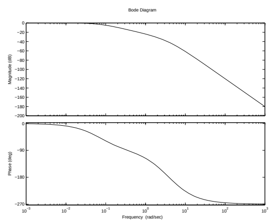
Open system A



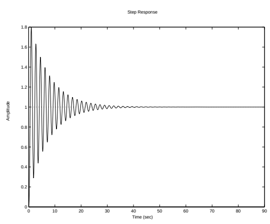
Open system B



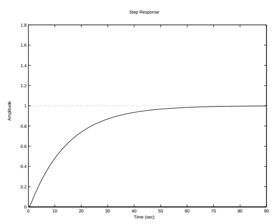
Closed loop system a



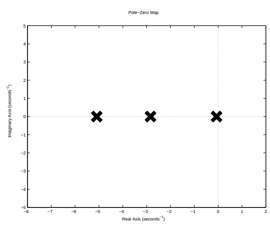
Closed loop system b



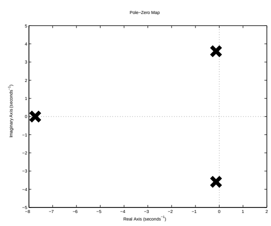
Step response 1



Step reponse 2



Poles I



Poles II

Figure 4.11a. The frequency scales of the Bode diagrams are the same. The time scales of the step responses are the same. In the pole-zero maps, $\times$ *marks poles.*

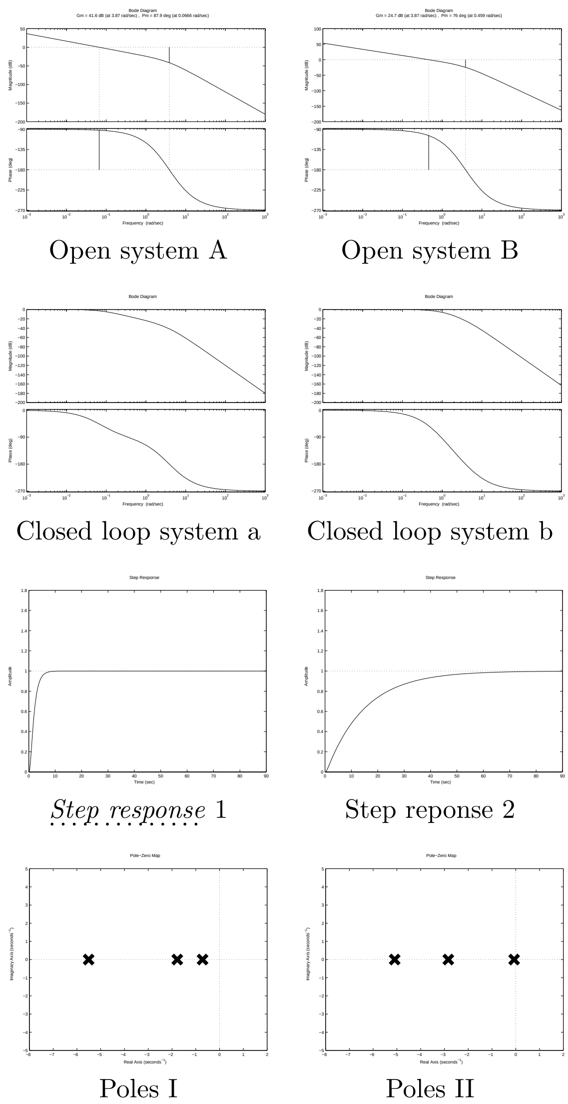


Figure 4.11b. The frequency scales of the Bode diagrams are the same. The time scales of the step responses are the same. In the pole-zero maps, $\times$ marks poles.

5 Compensation

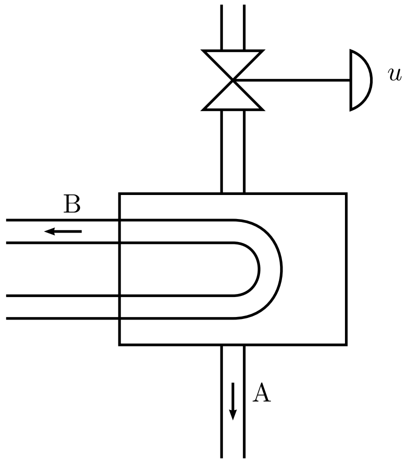


Figure 5.1a

5.1 The outflow temperature θ in the liquid A can be controlled in a heat exchanger by controlling the flow of the liquid B by a *valve* with the setting denoted u. See Figure 5.1a. Measurements have been made using a sinusoidal input u and the gain and phase shift have been measured at different frequencies. The results are given in the following table.

Frequency [rad/s]	Gain	Phase shift
0.05	1.37	$-67°$
0.1	0.80	$-106°$
0.2	0.34	$-153°$
0.3	0.18	$-185°$
0.4	0.11	$-210°$

a) Make a Bode plot for the system.

b) What is the largest *gain crossover frequency* possible to achieve when using proportional control and wanting a *phase margin* of at least 50°?

c) Suggest a compensator that doubles the speed compared to b) and still keeps the *phase margin*.

5.2 In Figure 5.2a we have arranged *step responses* and open loop and closed loop (feedback with -1) Bode plots for five different systems. Identify the three plots that belong to each of the five systems, one open loop and one closed loop Bode plot and one *step response*. Motivate your answer by pointing out one unique feature for each system.

Hint Answer Solution

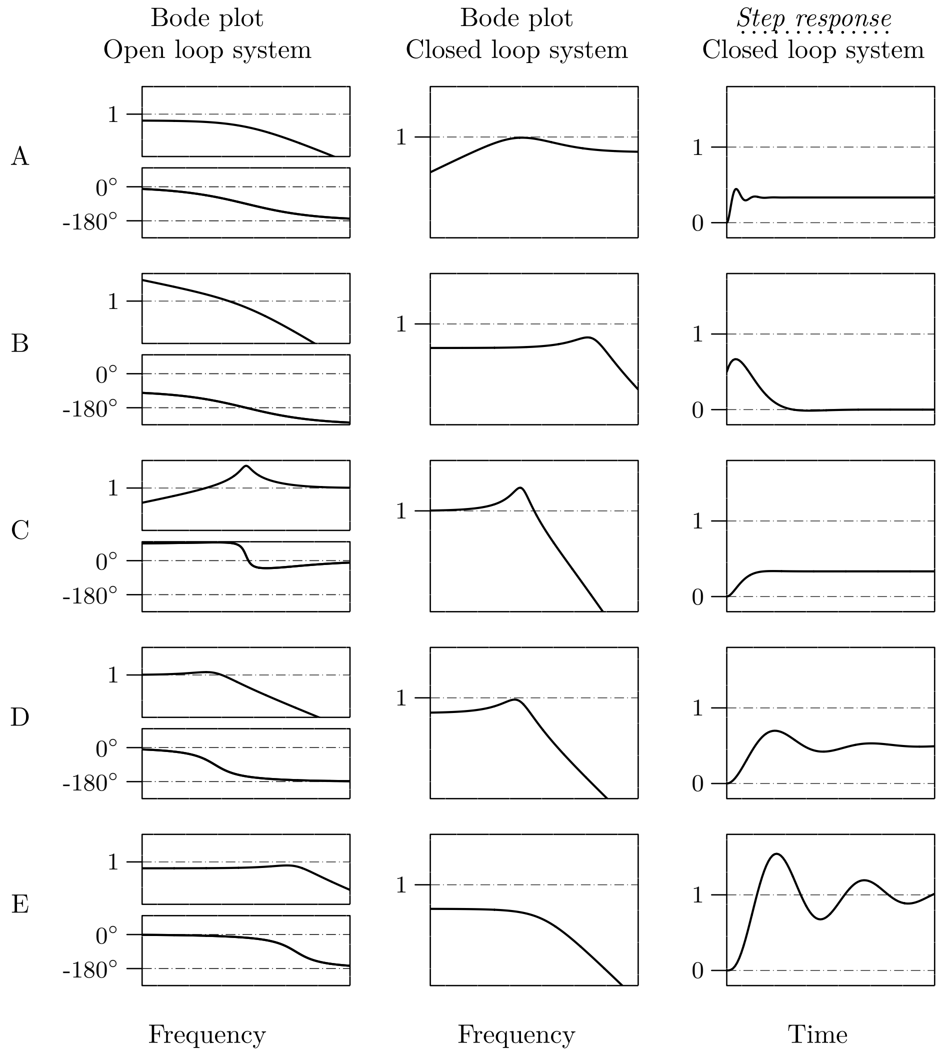


Figure 5.2a. All comparable diagrams have equal scaling.

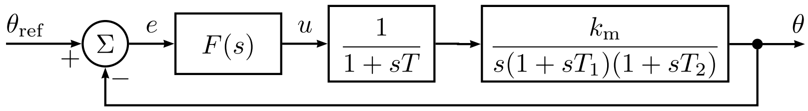


Figure 5.3a

5.3 A DC-servo is described by the block diagram in Figure 5.3a, where $T_1 = 50$ ms is a mechanical time constant, $k_{\mathrm{m}} = 10$ is a proportional constant, $T_2 = 25$ ms is an electrical time constant, and $T = 10$ ms is an amplifier time constant. The system is tested with $F(s) = 1$ and we find that the dynamic properties are satisfactory but that the system is somewhat too slow. Find an $F(s)$ so that the closed loop system is twice as fast as for $F(s) = 1$, without increasing the overshoot. $F(s)$ should also give a closed loop system which fulfills the following accuracy demands:

- $|\theta - \theta_{\mathrm{ref}}| \leq 0.001$ rad in steady state when θ_{ref} is constant.
- When θ_{ref} is a ramp with slope 10 rad/s we should have $|\theta - \theta_{\mathrm{ref}}| \leq 0.01$ rad in steady state.

Hint Answer Solution

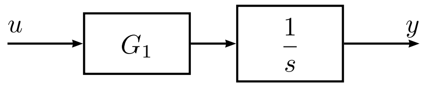


Figure 5.4a

5.4 A system $G(s)$ can be split into two sub-systems

$$G(s) = G_1(s)\frac{1}{s}$$

according to Figure 5.4a. The Bode plot for $G_1(s)$ is given in Figure 5.4b. Find a compensator for the system $G(s)$ such that the following is fulfilled:

- The *phase margin* for the compensated system is 40°.
- The closed loop system is twice as fast as what is possible to achieve using proportional control with a 40° *phase margin*.
- The *steady state error* when the reference signal is a ramp is 1% of the corresponding error with proportional control and 40° *phase margin*.

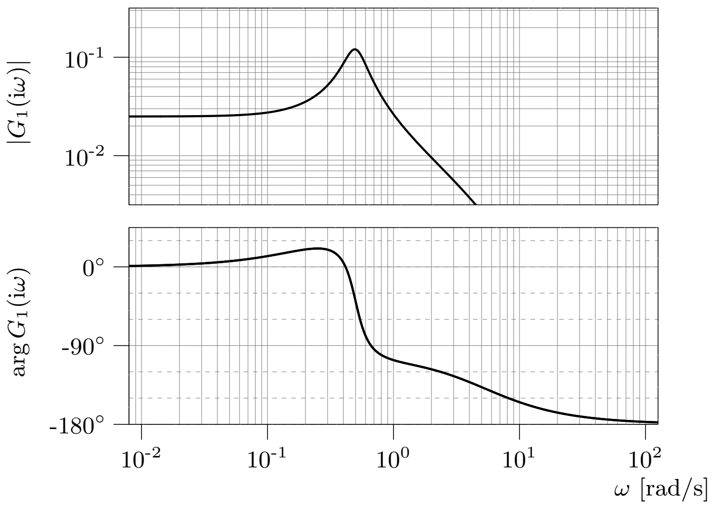


Figure 5.4b

Hint Answer Solution

5.5 The Bode plot for a system $G(s)$ is given in Figure 5.5a.

a) What is the *gain crossover frequency* ω_c, *phase margin* ϕ_m, *phase crossover frequency* ω_p, and *gain margin* A_m of the system? Is the closed loop $\frac{G(s)}{1+G(s)}$ stable?

b) Assume that the system is controlled using the proportional feedback

$$U(s) = K(R(s) - Y(s))$$

For which $K > 0$ is the closed loop system asymptotically stable?

c) Assume that we choose $K = 2$ in the proportional controller in problem 5.5 b). What will the *steady state error* be when $r(t) = 10t$?

d) Assume that $y(t)$ is delayed T seconds. How large is T allowed to be in order for the system to still be asymptotically stable with $K = 2$?

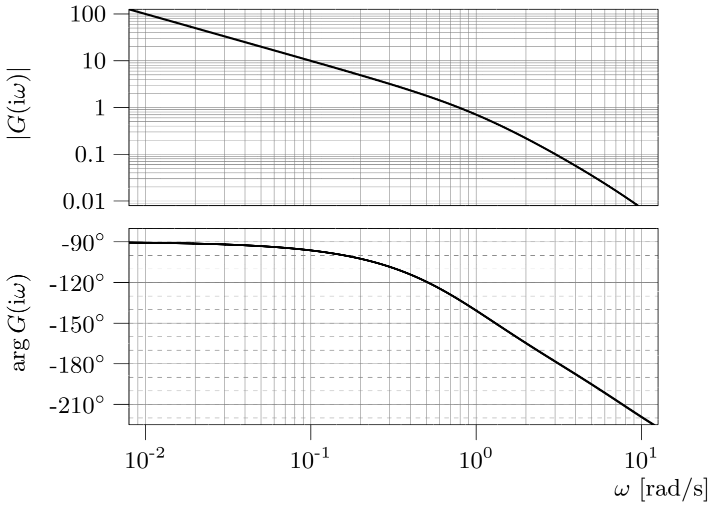


Figure 5.5a

e) Draw the Nyquist curve of the system.

Hint Answer Solution

5.6 a) A plot of the amplitude curve of a stable transfer function $G_o(s)$ is given in Figure 5.6a. Choose one of the following alternatives regarding the stability of the closed loop system $\dfrac{G_o}{1+G_o}$:

1. It is stable.
2. It is not stable.
3. Impossible to determine given these facts only.

b) Repeat for the transfer function whose amplitude curve is given in Figure 5.6b. Justify your answers carefully.

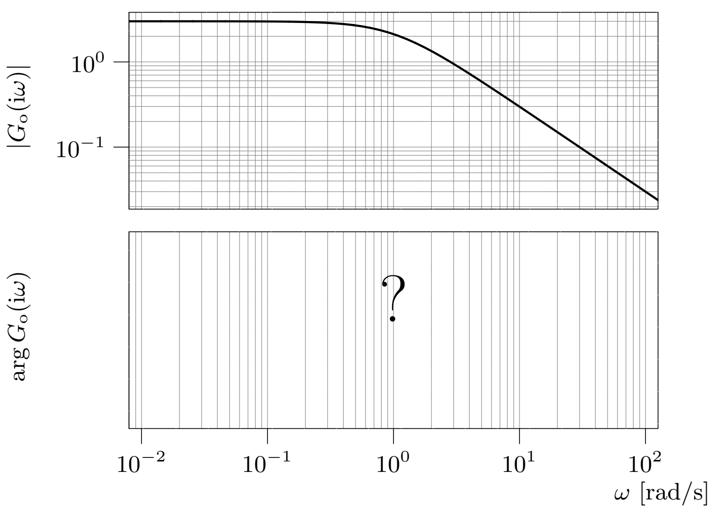


Figure 5.6a

Hint Answer Solution

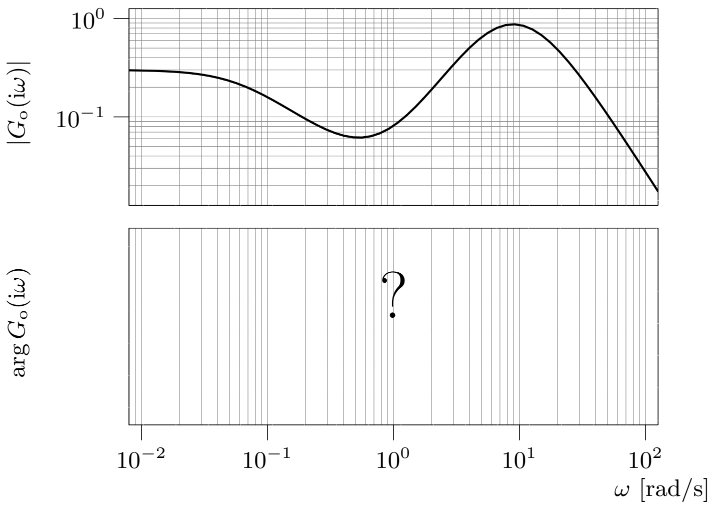


Figure 5.6b

5.7 Consider a system described by

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

a) Assume that the system is controlled by

$$U(s) = F(s)(R(s) - Y(s))$$

where $F(s) = 1$. Find ω_c, ω_p, φ_m, and A_m for the *loop gain*.

b) Compute a regulator such that the open loop system fulfills the following requirements:
   (i) $\omega_c = 5$
   (ii) $\varphi_m \geq 60°$

   and the closed loop system fulfills:
   (iii) $e_0 = 0$

   Draw the Bode plot of the compensated open loop system and check that the requirements are satisfied. Simulate the closed loop system for a step in the reference signal and plot the *step response*. Check that the requirement on the *steady state error* is satisfied.

c) Draw the amplitude curve of the Bode plot of the closed loop system with and without the compensator. Describe how the properties of the closed loop system have been changed by the compensation.

d) Simulate the control error when the reference signal is a ramp and the regulator designed in b) is used. Is the stationary error zero?

Hint Answer Solution

5.8 A vehicle is described by the model

$$Y(s) = G(s)U(s)$$

where u is the input and y is the position and

$$G(s) = \frac{0.1}{s(s+1)^2}$$

The Bode diagram of the model is given in Figure 5.8a.

a) Assume that the vehicle is controlled using proportional feedback

$$U(s) = K(R(s) - Y(s))$$

What is highest *gain crossover frequency* that can be achieved if it is required that the *phase margin* is at least 60°? For which value of K is this*gain crossover frequency* obtained?

b) Assume that it is required that the vehicle is able to follow a reference path given by the function

$$r(t) = 0.5 \cdot t \quad t \geq 0$$

What is the resulting *steady state error* if the vehicle is controlled by the proportional *feedback* designed in a).

c) Design a controller,

$$U(s) = F(s)(R(s) - Y(s))$$

for the vehicle above, such that the resulting control system fulfills the following requirements:

- ◇ The *steady state error*, using the same reference signal as in b), is less that 10% of what was achieved in problem b).
- ◇ The *phase margin* is at least 60°.
- ◇ The *gain crossover frequency* is the same as what was obtained in problem a).

Hint Answer Solution

5.9 The upper figure in Figure 5.9a shows, in four different cases, the Bode diagram for the *loop gain*

$$G_O(i\omega) = F(i\omega)G(i\omega)$$

and the lower figure in Figure 5.9a shows the amplitude curve, $\mid G_C(i\omega) \mid$, for the *transfer function* of the closed loop system

$$G_C(s) = \frac{G_O(s)}{1 + G_O(s)}$$

in the corresponding four cases. Combine the diagrams in Figure 5.9a.

Hint Answer Solution

5.10 A system is described by the relationship

$$Y(s) = G(s)U(s)$$

where the Bode diagram of $G(s)$ is given in Figure 5.10a.

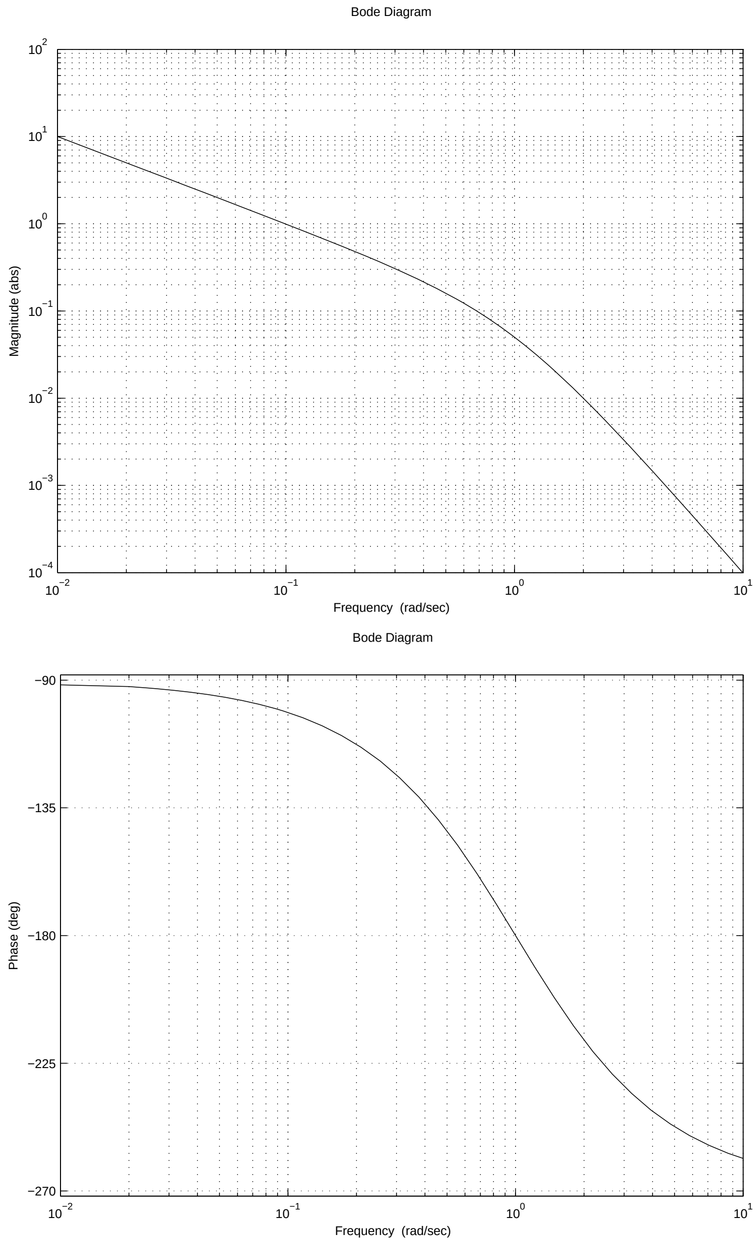


Figure 5.8a

a) Give a possible combination of values of p, n and m such that the diagram corresponds to the *transfer function*

$$G(s) = \frac{K(s+z_1)\cdots(s+z_m)}{s^p(s+p_1)\cdots(s+p_n)}$$

b) Assume that the system is going to be controlled using the feedback

$$U(s) = F(s)(R(s) - Y(s))$$

Determine $F(s)$ such that the control system fulfulls the following requirements:

- ◇ The *steady state error* is zero when the reference signal is a *unit step*.

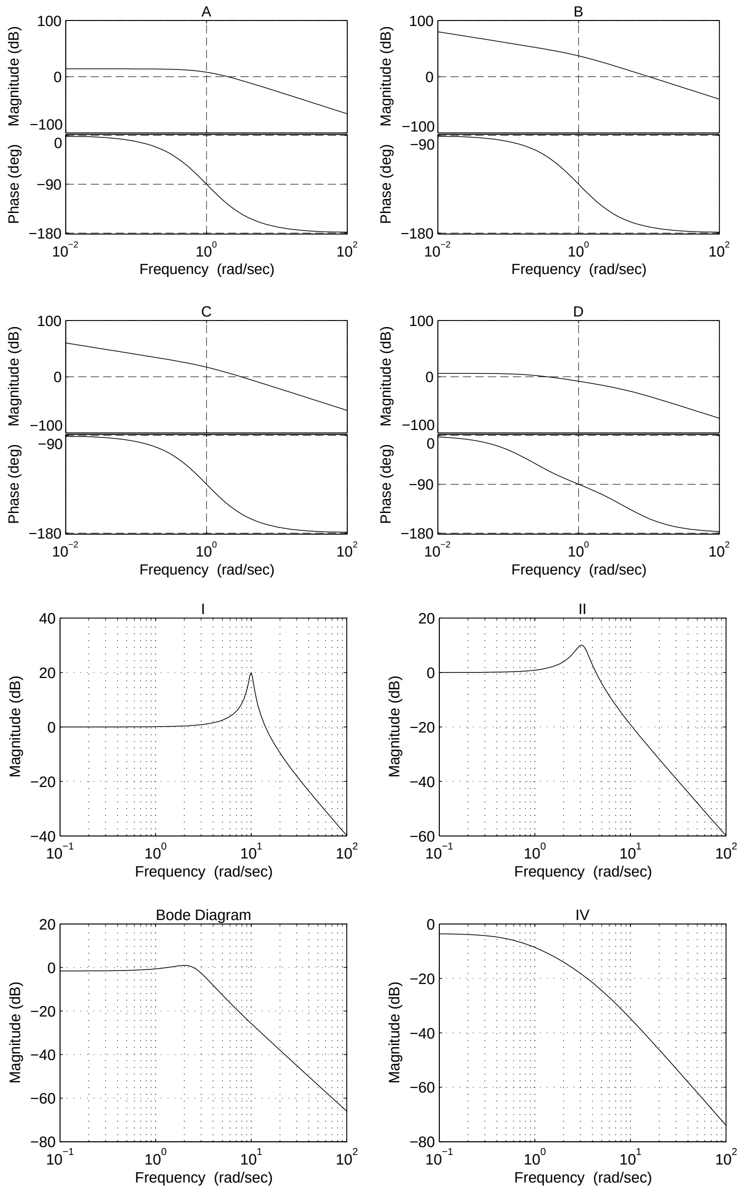


Figure 5.9a. Upper: $G_O(i\omega)$. *Lower:* $\mid G_C(i\omega) \mid$.

- ◇ The absolute value of the *steady state error* is less than 0.01 when the reference signal is a unit ramp.
- ◇ The compensated open loop system has *gain crossover frequency* 3 rad/s and *phase margin* 45°.

Hint Answer Solution

5.11 A system is described by the model

$$Y(s) = \frac{2}{(10s+1)^2} U(s)$$

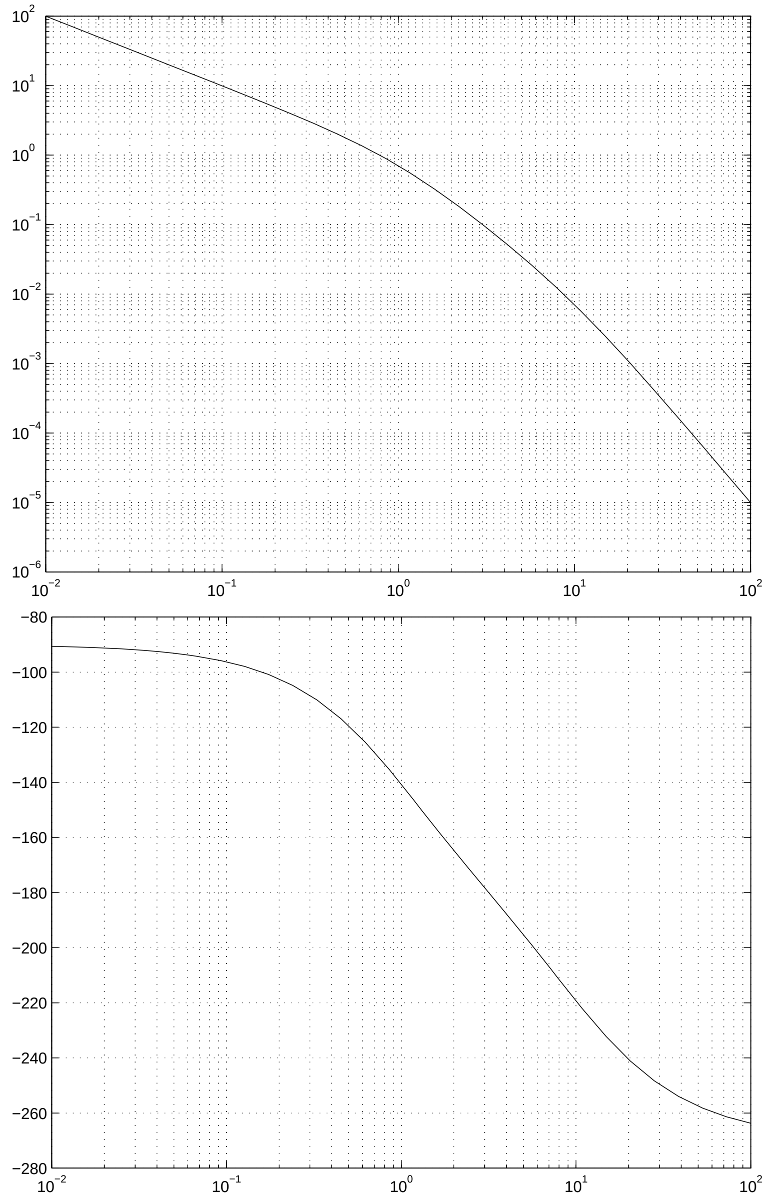


Figure 5.10a

and is controlled by the feedback

$$U(s) = K\frac{\tau_{\mathrm{D}}s+1}{\beta\tau_{\mathrm{D}}s+1}(R(s)-Y(s))$$

In Figure 5.11a, four *step responses* for the parameter pairs

$$\begin{aligned} i) &\quad K=10 \quad \beta=0.1\\ ii) &\quad K=10 \quad \beta=0.8\\ iii) &\quad K=5 \quad \beta=0.1\\ iv) &\quad K=5 \quad \beta=0.8 \end{aligned}$$

are shown. Combine these values with the *step responses* in Figure 5.11a.

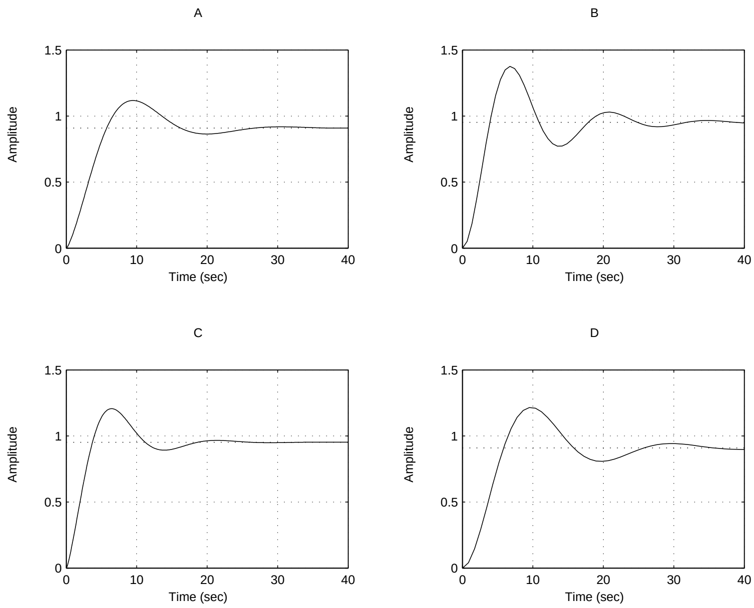


Figure 5.11a

6 Sensitivity and Robustness

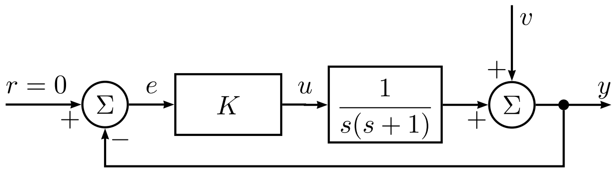


Figure 6.1a

6.1 Consider the control system in Figure 6.1a where $v(t)$ is a sinusoidal *disturbance*, $v(t) = \sin(t)$. Compute the absolute value of the *sensitivity function* at $\omega = 1$ rad/s as a function of K. How must K be selected if the amplitude of $y(t)$ shall be less than the amplitude of $v(t)$ at this frequency?

Hint Answer Solution

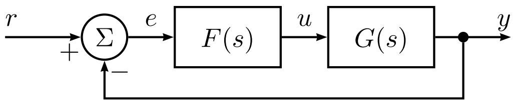


Figure 6.2a

6.2 Assume that we have constructed a controller $F(s)$ for the model $G(s)$, see Figure 6.2a, such that there is no *steady state error* when the reference signal is a step. Let the real system be given by

$$G^0(s) = (s + 1)G(s)$$

and assume that $G^0(s) \to 0, \ s \to \infty$. Also assume that the amplitude curve of the closed loop system has no resonance peaks and decreases, at least and asymptotically, with 20 dB_{20}/decade for frequencies over the bandwidth. What is the highest possible bandwidth we can use for the closed loop system in Figure 6.2a, while at the same time guaranteeing stability?

Hint Answer Solution

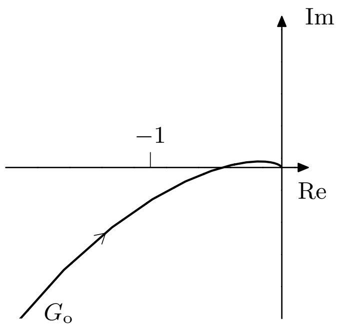


Figure 6.3a

6.3 Figure 6.3a shows a Nyquist diagram for the *loop gain* G_o. Show in a figure for what frequencies (that is, for what part of the Nyquist curve above) additive *disturbances* on the output are amplified in the sense that the output amplitude of the control system in Figure 6.3b is larger than the *disturbance* amplitude.

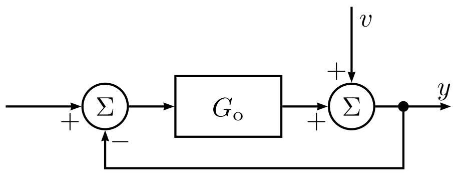


Figure 6.3b

Hint **Answer** **Solution**

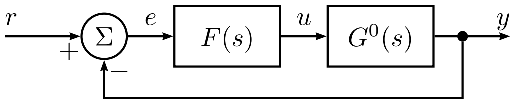


Figure 6.4a

6.4 Consider the control system in Figure 6.4a. The true system, denoted $G^0(s)$, is modeled as

$$G(s) = \frac{1}{s+10}$$

The controller

$$F(s) = \frac{s+10}{s}$$

gives an asymptotically stable closed loop system with the model $G(s)$. Now assume that the system is given by

$$G^0(s) = G(s)(1 + \Delta(s))$$

where it is known that $\Delta(s)$ has no poles in the right half plane, and that

$$|\Delta(\mathrm{i}\omega)| < \frac{0.9}{\sqrt{1+\omega^2}}$$

Can we be sure that the closed loop system is asymptotically stable?

Hint **Answer** **Solution**

6.5 A process is described by the model $G(s)$, while the process in reality has the *transfer function*

$$G^0(s) = e^{-sT} G(s)$$

a) Draw the absolute value of the inverse of the relative model error, that is,

$$\frac{1}{|\Delta(\mathrm{i}\omega)|}$$

b) Assume that we design a controller $F(s)$ starting with the model $G(s)$. How large may

$$\left| \frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1 + F(\mathrm{i}\omega)G(\mathrm{i}\omega)} \right|$$

be at most, in order to guarantee asymptotic stability of the closed loop system for all values of T, when the controller $F(s)$ is used on the system $G^0(s)$?

Hint **Answer** **Solution**

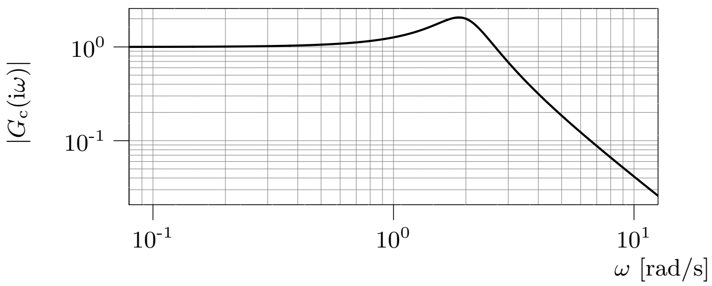


Figure 6.6a

6.6 A DC-motor is assumed to have the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

and it is controlled using proportional feedback,

$$U(s) = F(s)(R(s) - Y(s))$$

where $F(s) = 4$. The amplitude curve of the feedback system

$$|G_c(i\omega)| = \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

is given in Figure 6.6a. Assume that the real system is given by

$$G^0(s) = G(s)\frac{\alpha}{s+\alpha}, \quad \alpha > 0$$

and the controller $F(s)$ is used on the system $G^0(s)$.

a) Draw a root locus with respect to α for the characteristic equation of the closed loop system and determine for which α the closed loop system is asymptotically stable.

b) Use the robustness criterion to decide for which α the closed loop system is asymptotically stable.

c) Comment on the possible differences in the demands on α in a) and b).

Hint Answer Solution

6.7 A system $G^0(s)$ is controlled using a regulator $F(s)$. In Figure 6.7a the amplitude part the Bode plot of the nominal closed loop system,

$$G_c(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

is shown. It is known that G_c is stable, and it is assumed that G and G^0 have the same number of poles in the right half plane. The model uncertainty $\Delta(s)$, defined by

$$\Delta = \frac{G^0 - G}{G}$$

is assumed bounded by $|\Delta(i\omega)| \leq \gamma\omega$. In what interval must γ lie to guarantee stability of the closed loop system?

Hint Answer Solution

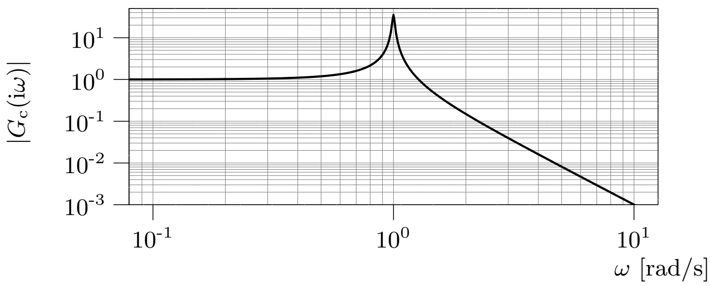


Figure 6.7a

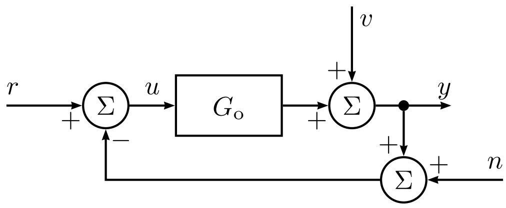


Figure 6.8a

6.8 Consider the system in Figure 6.8a. For $r(t) = 0$, $n(t) = 0$ and $v(t) = \sin t$ the steady-state output is given by

$$y(t) = \frac{1}{\sqrt{2}} \sin(t - \frac{\pi}{4})$$

Determine the steady-state output $y(t)$ when $r(t) = 0$, $v(t) = 0$ and $n(t) = \sin t$.

Hint Answer Solution

6.9 Assume we are have worked with a nominal model

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

and developed a controller (PID with low-pass filtered derivative)

$$F(s) = 0.46 \cdot \frac{0.43s+1}{0.090s+1} \cdot \frac{2.0s+1}{2.0s}$$

Assume that the true system contains a time constant that was neglected, and that the *transfer function* of the system is given by

$$G^0(s) = G(s)\frac{1}{s+1}$$

a) Determine the relative model error $\Delta(s)$.

b) Draw $\frac{1}{|\Delta(i\omega)|}$ and $\left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right|$ in a Bode plot, for the two cases $F(s) = 1$ and $F(s)$ given above. What can be said about the robustness of the closed loop system in these two cases when $F(s)$ is used for control of the true system $G^0(s)$?

Hint Answer Solution

7 Special Controller Structures

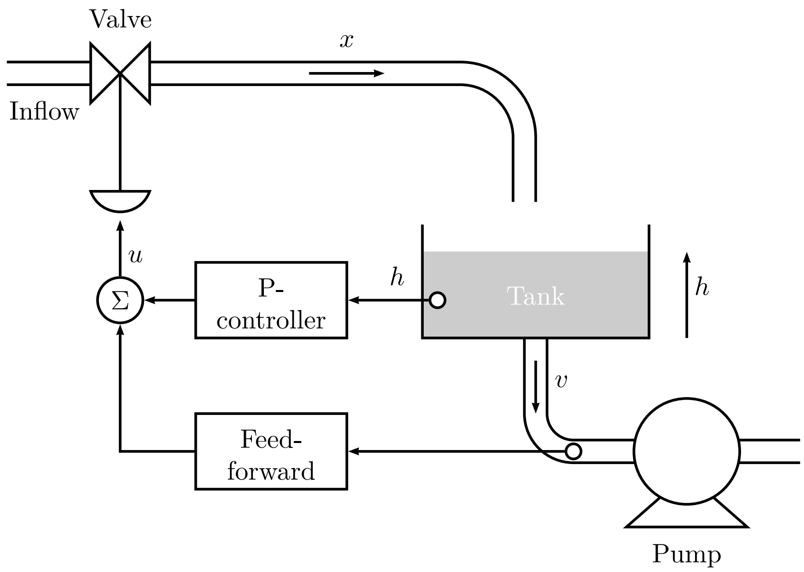


Figure 7.1a

7.1 A level control system for a water tank is shown in Figure 7.1a, where all variables denote offsets from an operation point. The inflow, $x(t)$, to the tank (with area A) is determined by the *valve*, and the outflow, $v(t)$, is determined by the pump. Stu Dent has got the assignment to keep the water level in the tank constant, in spite of variations in the outflow $v(t)$. First, Stu determines the *transfer function* $G_{\mathrm{v}}(s)$ from the *valve* to $x(t)$. By *step response* experiments, he obtains the following result:

$$G_{\mathrm{v}}(s) = \frac{1}{0.5s + 1}$$

a) Because the *disturbance* $v(t)$ is measurable, Stu first considers a feedforward compensator to completely eliminate it. Stu, who also knows that it is dangerous to differentiate the *disturbance*, cancels all the derivative terms in the compensator. Compute the feedforward compensator, and determine the response $h(t)$ Stu will get, if the outflow $v(t)$ is changed stepwise with an amplitude of 0.1.

b) To improve the control system, Stu also introduces a proportional *feedback* of the water level h. What is the steady-state error in the level h now, if the outflow is changed in the same way as in a)?

Hint Answer Solution

7.2 Consider the following system

$$Y(s) = \frac{2}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where u is the control signal, y is the output and v is a *disturbance*. It is desired that y should be as small as possible despite the *disturbance* v.

a) Design a feedforward controller from v to u that eliminates the influence of v on y.

b) Assume that v is a pure sinusoid with amplitude 2. How large will the control signal be?

c) The real system is described by

$$Y(s) = \frac{b}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where b value is not exactly known but has its value close to 2. To solve this problem a P controller is added to the feedforward controller that was designed in a). The full controller looks like

$$U(s) = -KY(s) + F_{\mathrm{f}}(s)V(s)$$

where $F_{\mathrm{f}}(s)$ is the feedforward controller. What is the stationary error if $v = 1$?

Hint Answer Solution

7.3 The *transfer function* for a temperature control system is given by

$$Y(s) = \frac{3}{s+1}U(s) + \frac{4}{(s+2)(s+5)}V(s)$$

where y is the controlled temperature, u is the supplied power and v is the temperature of the surroundings. Assume that the desired temperature is zero.

a) Design a feedforward controller $U(s) = F_{\text{f}}(s)V(s)$ which eliminates the influence of the *disturbance* v on y.

b) To simplify implementation $F_{\text{f}}(s)$ is replaced with a constant, $\widetilde{F}_{\text{f}} = F_{\text{f}}(0)$. Assume that v is given by $v(t) = -1 - 0.1t$ and that $U(s) = \widetilde{F}_{\text{f}}V(s)$ is used. What will $y(t)$ be in steady state?

c) The previous controller is now extended with a P controller:

$$U(s) = \widetilde{F}_{\text{f}}V(s) - KY(s)$$

What will now $y(t)$ be in steady state?

d) Assume that one only uses the P controller

$$U(s) = -KY(s)$$

What will now $y(t)$ be in steady state?

Hint Answer Solution

8 State Space Description

8.1 Define suitable state space variables for the DC motor discussed in Problem 2.1, and write the system in state space form.

Hint Answer Solution

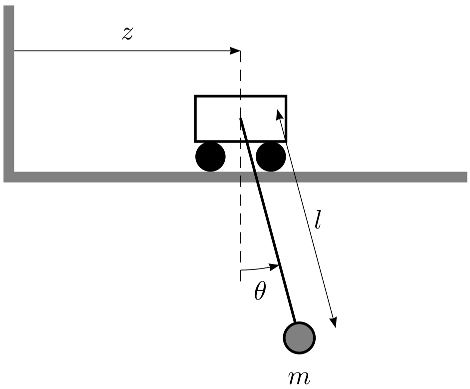


Figure 8.2a

8.2 Consider the system illustrated in Figure 8.2a. It consists of a hinge that can move in the direction marked "z", and an attached pendulum. The system is described by the equation

$$\ell\ddot{\theta} + g\sin\theta + \ddot{z}\cos\theta = 0$$

Define state space variables, input, and output as

$$x_1 = \theta \quad x_2 = \dot{\theta} \quad u = \ddot{z}/\ell \quad y = \theta$$

and

$$\omega_0^2 = g/\ell$$

Linearize the system around the *equilibrium* point given by

$$x_1 = \pi \quad x_2 = 0 \quad u = 0$$

Hint Answer Solution

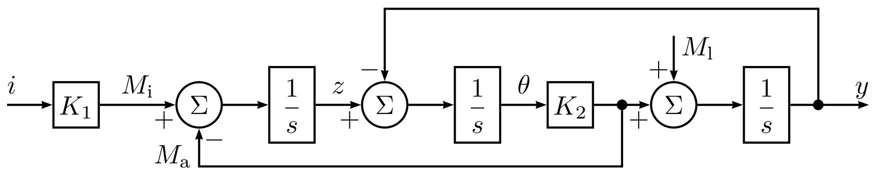


Figure 8.3a

8.3 The block diagram in Figure 8.3a describes an electric motor that drives a load via an elastic axis. Here i is the driving *current* to the motor, which gives the *torque* M_i. z is the turning rate of the motor and y is the turning rate of the load. θ is the angle of the transmission axis. $M_\mathrm{a} = K_2\theta$ is the *torque* this angle causes. M_l is the *torque* from the load. Give a state space description for the system with M_l and i as inputs and y as output. (There are at least two different ways to solve this problem.)

Hint Answer Solution

8.4 Write the following systems in state space form.

a)

$$G(s) = \frac{2s+3}{s^2+5s+6}$$

b)

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + 6\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 11\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 6u$$

c)

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = 4\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\mathrm{d}u}{\mathrm{d}t} + 2u$$

Use for example *controllable* or observable canonical form or try to perform a realization manually using e.g. diagonal form or by (cleverly) guessing the states.

Hint Answer Solution

8.5 A system has the impulse response (weight function)

$$g(t) = 2e^{-t} + 3e^{-4t}$$

Write the system in state space form.

Hint Answer Solution

8.6 Consider the system

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} -1 & 2 \end{pmatrix} x$$

Compute the *transfer function* of the system.

Hint Answer Solution

8.7 A (bad) *state space model* of

$$G(s) = \frac{1}{s+1}$$

is given by

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

a) Compute $x_1(t)$ $x_2(t)$ and $y(t)$ if $x(0) = 0$ and

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

b) Is the system asymptotically stable? Input-output-stable?

c) Examine the controllability and observability for the system.

d) Explain why the realization is not suitable for simulating a system whose transfer function is $G(s)$.

Hint Answer Solution

8.8 Compute the poles and zeros of the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Hint Answer Solution

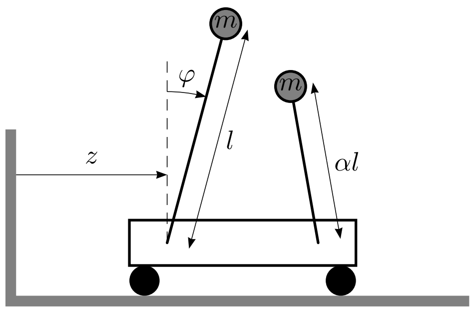


Figure 8.9a

8.9 Two mathematical pendulums are mounted on a trolley. They are mounted so that they can move without friction in a plane coinciding with the direction of movement for the trolley. The lengths of the pendulums are ℓ and $\alpha\ell$ and their masses are m. For one pendulum we have

$$\ddot{z}\cos\varphi + \ddot{\varphi}\ell = g\sin\varphi$$

a) Linearize the system around $\varphi = 0$ and put the constants ℓ, m, and g to 1 and write the equations in the form $\dot{x} = Ax + Bu$.

b) Give the values on α for which the system is controllable. Give a practical motivation.

Hint Answer Solution

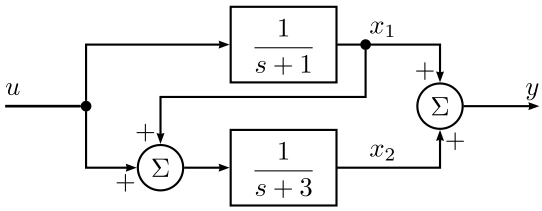


Figure 8.10a

8.10 A system is given by the block diagram in Figure 8.10a. Derive a state space model of the system, with the state space variables given in the figure.

Hint Answer Solution

8.11 An electromagnet can make a metal ball levitate. The electromagnet is positioned so as to make the magnetic force act upwards in the vertical plane. By placing a metal ball under the magnet, the ball stays in the air if the magnetic force exactly matches the gravity acting on the ball.

The gravitational force is $F_g = mg$ where m in the mass of the ball and g is the gravitational constant. The force from the magnet is described by $F_m(t) = ku(t)y^{-2}(t)$ where $u(t)$ is the current through the coil of the electromagnet, $y(t)$ is the distance from the magnet to the ball and k is a proportionality constant. Force equilibrium leads to $m\ddot{y}(t) = -\frac{ku(t)}{y^2(t)} + mg$.

Using the states $x_1 = y$ and $x_2 = \dot{y}$ leads to the state space description

$$\dot{x}(t) = \begin{pmatrix} x_2(t) \\ -\frac{ku(t)}{mx_1^2(t)} + g \end{pmatrix}$$
$$y(t) = x_1(t)$$

a) Show that all stationary points are given by $x_1 = x_{10}$, $x_2 = 0$ and $u_0 = g\frac{mx_{10}^2}{k}$.

b) Determine a linear state space model approximating the nonlinear system around each stationary point.

Hint Answer Solution

8.12 A simplified description of a system for controlling the water level in a dam is shown in Figure 8.12a.

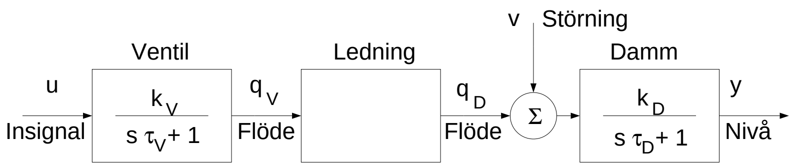


Figure 8.12a. Dam system

Assume that the influence of the pipe can be neglected, i.e. $q(t) = q_V(t) = q_D(t)$. Verify that the system, using the state variables $x_1(t) = y(t)$ and $x_2(t) = q(t)$ can be expressed in state space form as

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{\tau_D} & \frac{k_D}{\tau_D} \\ 0 & -\frac{1}{\tau_V} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{k_V}{\tau_V} \end{pmatrix} u(t) + \begin{pmatrix} \frac{k_D}{\tau_D} \\ 0 \end{pmatrix} v(t) \qquad y(t) = (1 \;\; 0)x(t)$$

Hint Answer Solution

8.13 An object with mass m moves horizontally with non-negative velocity $x(t)$ subject to a driving force $u(t)$ and *air drag*, which is proportional to the square of the velocity. The object can hence be described by the differential equation

$$m\dot{x}(t) = -cx^2(t) + u(t)$$

where the coefficient c describes the aerodynamic properties of the object.

a) Assume the force is constant $u(t) = u_0$. Determine the steady state velocity (equilibrium) of the object. How can it be seen that this is a non-linear system? How much must the force increase in order to double the velocity?

b) Linearize the system around the stationary point (x_o, u_0) and formulate a linear system using the variables

$$\Delta x(t) = x(t) - x_0 \quad \Delta u(t) = u(t) - u_0$$

c) Determine the pole of the linearized system. Which factors affect the location of the pole? How does the force in the stationary point affect the location of the pole? Does that seem natural considering the "shape" of the right hand side of the state equation?

Hint Answer Solution

9 State Feedback

9.1 Consider the system

$$\dot{x} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

a) Calculate state *feedback* laws $u = -Lx + r$ that places the poles in

I) $\{-3, -5\}$,

II) $\{-10, -15\}$

What limits the possibility to achieve arbitrarily fast dynamics of the real closed loop system?

b) Extend the controllers to include a feedforward scaling l_0 on the reference, $u = -Lx + l_0 r$. Derive the closed-loop *transfer function* from r to y for the control law in (I). Based on the result, suggest a value on l_0 such that a constant reference signal can be followed without any stationary error.

Hint Answer Solution

9.2 A system can be described in state space form as

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & -1 \end{pmatrix} x$$

We want to place the closed loop poles in $\{-2, -3\}$. Create a controller based on pole placement and *feedback* of observer states.

Hint Answer Solution

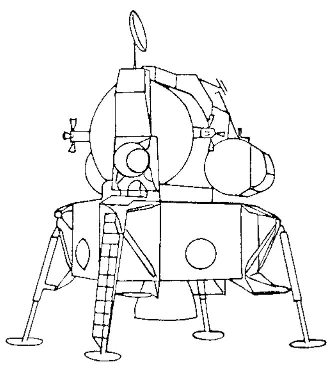


Figure 9.3a

9.3 Figure 9.3a shows the Lunar Excursion Module from the Apollo project. Consider the module hovering a short distance above the surface of the moon using its main engine. If the pitch angle of the module (angle between the vertical line and the direction of movement) differs from zero, a horizontal component of the force is obtained and the module is accelerating along the surface.

We will study a block diagram which shows the connection between the input u (the control signal to the attitude thrusters), the pitch angle θ and the position coordinate z. See Figures 9.3b and 9.3c.

The module is both in the θ-direction and in the z-direction obeying Newton's law of motion without any kind of damping. The *transfer function* from the control signal of the astronaut (y_{ref}) to velocity $\dot{z}$ is

$$\frac{K_1 K_2}{s^3}$$

which is very difficult to control by hand.

a) Write the system in state space form.

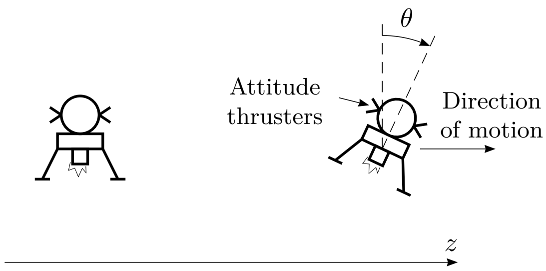


Figure 9.3b

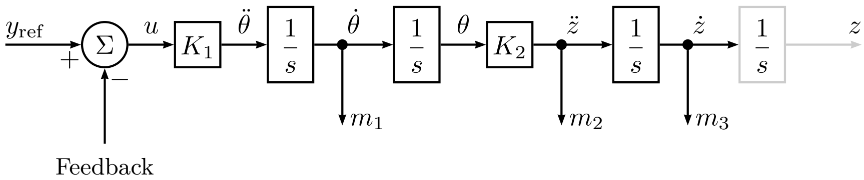


Figure 9.3c

b) In order to make the control duty of the astronaut easier we change the dynamics of the module by making internal *feedback*. The following signals are measurable:

m_1, the attitude angular velocity measured using rate gyro.

m_2, the acceleration in z-direction measured using accelerometers positioned on gyro-stabilized platforms.

m_3, the velocity in z-direction measured using doppler-radar.

Calculate a state-feedback using these signals such that the closed loop system obtains its poles in $s = -\frac{1}{2}$ and the control signal of the astronaut becomes the reference signal of the velocity in z-direction.

c) Suppose we by safety reasons are interested in the possibility of controlling the module even if the sensors measuring m_1 and m_2 are not working. Design a controller that can handle this and has approximately the same behavior as in a).

Hint Answer Solution

9.4 A DC motor with an external load, f, is described by

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{1}{\tau}\omega + c_1 u + c_2 f$$

where θ is the angle, ω the angular velocity, u the control signal, f the torque of the load, and c_1, c_2, and τ are constants.

a) Let $y = \theta$ and define x and the matrices A, B, C, H in a state-space representation $\dot{x} = Ax + Bu + Hf$, $y = Cx$.

b) Introduce a controller

$$u = l_0\theta_{\text{ref}} - l_1\theta - l_2\omega$$

such that the poles of the closed loop system becomes $\frac{1}{\tau}(-1 \pm \mathrm{i})$ and $\theta = \theta_{\text{ref}}$ in steady-state if $f = 0$ and θ_{ref} is constant.

c) Introduce a modified controller

$$u = l_0\theta_{\text{ref}} - l_1\theta - l_2\omega + u'$$

such that $\theta = \theta_{\text{ref}}$ in steady-state even for constant non-zero f and constant θ_{ref}.

Hint Answer Solution

9.5 Is it possible to design an observer with poles in $\{-5, -6, -7, -8\}$ for the system below? Motivate your answer.

$$\dot{x} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 10 \\ -3 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x$$

Hint Answer Solution

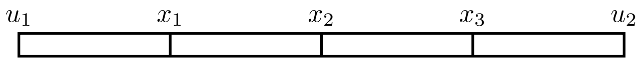


Figure 9.6a

9.6 We want to control the temperature in a long copper rod by heating or cooling its endpoints. Principally, this problem is described by a partial differential equation. To simplify the problem we assume that the temperature profile in the rod can be approximated by the temperatures x_1, x_2, and x_3 at three points. The temperatures in the end points are the inputs, u_1 and u_2. All temperatures are relative to the temperature of the surroundings.

We get the following ordinary differential equations:

$$\begin{aligned}\dot{x}_1 &= \alpha(u_1 - x_1) + \alpha(x_2 - x_1)\\ \dot{x}_2 &= \alpha(x_1 - x_2) + \alpha(x_3 - x_2)\\ \dot{x}_3 &= \alpha(x_2 - x_3) + \alpha(u_2 - x_3)\end{aligned}$$

where α is a constant that depends on the coefficient of thermal conductivity and the specific heat of the rod. For simplicity, let $\alpha = 1$. Consider the problem of controlling the temperature in x_1, x_2, and x_3 with u_1 only, assuming $u_2 = 0$.

a) Assume that we want to have an *arbitrary* temperature profile, that is, reach *arbitrary* values of x_1, x_2, and x_3. Is this possible? Why/why not?

b) Assume that all the temperatures x_1, x_2 and x_3 are measurable. Find a state *feedback* that brings any initial state to zero as e^{-3t}.

c) Assume that only one of the temperatures x_1, x_2, or x_3 is measurable, and that we still want a controller which damps a *disturbance* as e^{-3t} by using an observer. The sensor can be placed so that any of the three values x_1, x_2, or x_3 is measured. Which choices of measure point make it possible to control the system as desired? Give a motivation. Choose one of the points making the wanted design possible and design a controller, that is, an observer and a state feedback, giving the desired error damping.

Hint Answer Solution

9.7 Consider the model of a *DC-motor* from applied voltage to angle.

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{1}{s(s+1)}$$

a) Generate the *transfer function* in MATLAB and convert it to a *state space model* using `ss`. Which physical signals are represented by the states in the model that `ss` creates from the *transfer function*?

b) Suppose that the system is going to be controlled using state feedback

$$u(t) = -Lx(t) + l_0 r(t)$$

Compute the gain vector L using the command `place` for the following two choices of closed loop poles:

- ◇ Real poles at $\{-2.2, -2.1\}$
- ◇ Poles at $-1 \pm \mathrm{i}$

Create a temporary ss object of the closed loop system from r to y with l_0 set to 1, and use the command dcgain to compute the resulting closed loop static gain, then select l_0 based on this information to ensure that the final closed loop system has static gain 1. With $u = -Lx + l_0 r$ our closed-loop system is given by

$$\dot{x}(t) = (A - BL)x(t) + Bl_0 r(t)$$
$$y(t) = Cx(t)$$

Note that this is the system $C(sI - (A - BL))^{-1}Bl_0$, and that the static gain of this system is linear in l_0. Hence, if the static gain is 25 when l_0 is 1, it means we should use $l_0 = 1/25$ to achive unit static gain.

Using the ss command the closed-loop system is created with

```
G_r_to_y = ss(G.A-G.B*L,G.B*l_0,G.C,0);
```

What is the trade-off between response speed and control signal magnitude? A simple way to look at the control signal when performing a step is to define a linear system which outputs the control law $-Lx + l_0 r$ instead of the controlled signal y.

$$\dot{x}(t) = (A - BL)x(t) + Bl_0 r(t)$$
$$u(t) = -Lx(t) + l_0 r(t)$$

Compare this to a standard state-space model and we have

```
G_r_to_u = ss(G.A-G.B*L,G.B*l_0,-L,l_0);
step(G_r_to_y);
step(G_r_to_u);
```

c) Now let L be computed using linear quadratic optimization (LQ) in order to minimize the cost function

$$\int_0^\infty x(t)^{\mathrm{T}} Q x(t) + u(t)^2 \, \mathrm{d}t$$

for the three choices of weight matrices given below. Compute the closed loop poles and the step responses of the closed loop system for the three cases. Describe how the properties of the step responses in the different cases.

$$\text{(i)} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{(ii)} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$$
$$\text{(iii)} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$$

d) Start from case (ii) and increase the weight on the control signal gradually until the cost function becomes

$$\int_0^\infty x(t)^{\mathrm{T}} Q x(t) + 10\, u(t)^2 \, \mathrm{d}t$$

Compare the result with the result obtained for case (i).

e) Start from case (i) and introduce a weight on the velocity $\dot{y}(t)$. Increase the weight gradually and study how the poles and the step response of the closed loop system change.

Hint Answer Solution

9.8 The ingestion and metabolism of a drug in a human body can be described by the following equations:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = -aq(t) + u(t)$$
$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = aq(t) - bm(t)$$

where the input signal $u(t)$ is the ingestion rate of the drug, the output $y(t)$ is the mass $m(t)$ of the drug in the blood, and $q(t)$ is the mass of the drug in the gastrointestinal tract. The constants a and b are metabolism rates, satisfying $a > b > 0$. b characterizes the excretory process of the individual. In this example, $a = 0.05$ and $b = 0.02$.

a) Is the system *controllable*?

b) Design a state *feedback* that places the closed loop poles in -0.1.

$q(t)$ (the mass of the drug in the gastrointestinal tract) cannot be measured, so to be able to use the state *feedback* in b) we need an observer.

c) How should the poles of the observer be selected?

d) Design an observer with poles in -0.2.

Hint Answer Solution

9.9 An object with mass m moves, without friction, subject to a force $u(t)$ and can be described by the differential equation

$$m\ddot{y}(t) = u(t)$$

where $y(t)$ is the position of the object.

a) Let the state variables be chosen as $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, assume that $m = 1$ and write the model in state space form.

b) Determine a state feedback

$$u(t) = -Lx(t) + \tilde{r}(t)$$

where

$$L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}$$

such that the poles of the closed loop system have absolute value ω_0 and relative damping ζ.

c) Assume that $\tilde{r}(t) = l_0 r(t)$. Determine the transfer function from $r(t)$ to $y(t)$.

d) Assume that l_0 is chosen such that the closed loop system achieves static gain 1. Verify that this leads to $l_0 = l_1$ independent of where the closed loop poles are placed. Why is this a natural property?

Hint Answer Solution

9.10 Consider again the model of a *DC-motor* in Problem 9.7, i.e.

$$G(s) = \frac{1}{s(s+1)}$$

a) Create (or reuse from Problem 9.7) the state space model of the motor.

```
>> s=tf('s');
>> G=1/(s*(s+1));
>> Gss=ss(G)
```

Notice (by studying the created state-space model) that the way MATLAB generates the state space model from the transfer function (called realization) results in that the output (angle) is given by $x_2(t)$ and that $x_1(t)$ represents the angular velocity $\dot{y}(t)$.

b) Generate a time axis from 0 to 10 seconds, a square wave, i.e. a series of steps, with period time 2 seconds. Plot the input and check that it looks OK.

```
>> t=0:0.01:10;
>> u=square(pi*t)';
>> plot(t,u)
```

Simulate the DC-motor using this input, and plot the result. This can be done using the command `lsim` as in the MATLAB sequence below. The function `lsim` gives both output, time and state vector as result:

```
>> [y,t,x]=lsim(Gss,u,t);
>> plot(t,x)
```

Do the states $x_1(t)$ and $x_2(t)$ behave as one can expect? The state equations are given by (verify that by looking at the matrices `Gss.a` and `Gss.b`)

$$\dot{x}_1(t) = -x_1(t) + u(t) \qquad \dot{x}_2(t) = x_1(t)$$

This means that the system from input to angular velocity behaves as a first order system.

c) Assume now that the measurement of the output (angle) is affected by a measurement disturbance in terms of a sinusoid

$$y_m(t) = y(t) + 0.01\sin(20t)$$

Add the disturbance and plot the measured signal. Is the effect of the disturbance visible?

```
>> ym=y+0.01*sin(20*t);
>> plot(t,ym);
```

d) Also assume that we want to estimate the angular velocity by an approximate differentiation of the measured angle. This can be done as

$$\hat{V}_{est}(s) = F_{diff}(s)Y_m(s)$$

where

$$F_{diff}(s) = \frac{s}{\tau s + 1}$$

and τ is a suitably chosen time constant. "True" differentiation is not possible in reality, and hence the denominator $(\tau s + 1)$ has been included.

Computing the estimate of the angular velocity, and comparing the estimate with the true angular velocity can be done using the following MATLAB sequence:

```
>> Fdiff=s/(0.1*s+1);
>> vel_est=lsim(Fdiff,ym,t);
>> plot(t,vel_est,t,x(:,1))
```

Try some different values of the time constant τ in the range $0.01 - 1$ and compare the result. Is it possible to get a good estimate of the velocity? What happens with the estimate for low and high values of τ in the suggested interval?

e) Let us now use an observer to estimate the angular velocity. The ob- server is given by the equation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

which can be rewritten as

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Bu(t) + Ky(t) = (A - KC)\hat{x}(t) + \begin{pmatrix} B & K \end{pmatrix} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \tag{9.1}$$

The observer hence has $u(t)$ and $y(t)$ as inputs, and it delivers the estimates $\hat{x}(t)$ of the states of the system. Assuming that the system is given as an LTI-object in state space form as the variable `Gss`, which means that the matrices A, B and C of the state space description are given by `Gss.a, Gss.b` and `Gss.c` respectively, an estimate can be obtained as follows.

Determine the gain K of the observer by placing the observer poles, i.e. the eigenvalues of the matrix $A - KC$, at desired locations. As an example, we here choose to place them in -2.

```
>> K=acker(Gss.a',Gss.c',[-2 -2])';
```

Generate the observer as an LTI-object. Compare equation (9.1).

```
>> G_obs=ss(Gss.a-K*Gss.c,[Gss.b K],eye(2),zeros(2,2));
```

Estimate the states and compare the estimate with the true angular velocity.

```
>> [yhat,t,xhat]=lsim(G_obs,[u ym],t);
>> plot(t,xhat(:,1),t,x(:,1))
```

Plot also the estimation error

```
>> plot(t,x(:,1)-xhat(:,1))
```

Try some different locations of the observer poles and observe the difference in behavior of the estimate. Compare also the estimate of the angle and the measurement.

Hint Answer Solution

11 Implementation

11.1 If you discretize the controller

$$U(s) = KN(\frac{s+b}{s+bN})E(s)$$

with Tustin's formula you get a discrete-time controller of the form

$$u(t) = \beta_1 u(t-T) + \alpha_1 e(t) + \alpha_2 e(t-T)$$

What are the values of α_1, α_2, and β_2, if $T = 0.1$, $N = 10$, $b = 0.1$, and $K = 2$?

Hint **Answer** **Solution**

Hints

This version: 2025-08-21

1 Mathematics

1.1 Use the table on page 233, and linearity of the Laplace transform (rule A.5 on page 232).

Go back

1.2 a) Plug in what you know about $\dot{y}(t)$ for large t

b) Apply Laplace transforms and solve for $Y(s)$

c) Insert the Laplace transform of the step $u(t)$ to obtain $U(s)$, and then apply the theorem $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$

Go back

1.3 Start by writing using partial fractions, such as $\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

Go back

1.4 Apply Laplace transform to all terms in the differential equation, and use the table on page 233. For (a), if you have learned about *transfer functions*, you can alternatively proceed by deriving the *transfer function* from u to y, and use the results on page 234 which lists *step responses* $y(t)$, assuming a *unit step* input $U(s) = \frac{1}{s}$ and $Y(s) = F(s)U(s)$, given a *transfer function* $F(s)$.

Go back

1.5 Apply Laplace transform to both differential equations, eliminate $Z(s)$, and go back.

Go back

1.6 Compute the roots!

Go back

1.7 $z = |z|\, e^{i \arg(z)}$, $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$, $\arg(a + b\mathrm{i})$ can be computed from $\arctan(b/a)$ but you have understand in which orthant you are and compensate accordingly ($\arg(1+\mathrm{i})$ and $\arg(-1-\mathrm{i})$ leads to the same result but their argument differ), $|z_1 z_2| = |z_1|\,|z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, $|z_1/z_2| = |z_1|\,/\,|z_2|$, $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$, $e^{\mathrm{i}\phi} = \cos(\phi) + \mathrm{i}\sin(\phi)$

Go back

1.8 x in dB_{20} is given by $20 \log_{10} x$

Go back

1.9 What should $A^{-1}A$ be?

Go back

2 Dynamic Systems

2.1 Start with $J\ddot{\theta}(t) = -f\dot{\theta}(t) + M(t)$ and try to write $M(t)$ as a function of $\theta(t)$ and $u(t)$ by eliminating $i(t)$.

Go back

2.2 What is the relationship between the response of the system and the pole locations? Stability, oscillations, and speed are typical properties to look for.

Go back

2.3 a) Use the *final value theorem* to find the steady state gain, and calculate the time constant by estimating the time to reach 63% of the final value.

b) If it moves with a certain velocity, how long time does it take to reach a position further down?

Go back

2.4 Identify the coefficients ω_0 and ζ in the system description for the known and unknown system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Go back

2.5 What do you know about complex vs real poles? What happens if you have a pole in the origin? Can you find the exact *step response* if you have the *transfer function* $\frac{1}{s}$? If you have a zero in the origin, what happens when you use the *final value theorem* on a step response?

Go back

2.6 The important properties to study is how the absolute value of the poles relate to how fast the system is, and how the relation between real and imaginary part (i.e. the *argument*) relates to oscillations and *overshoot*s (and thus *settling time*).

Go back

2.7 You can easily compute the zero by hand, but also try using the command `tzero`.

Go back

2.8 See Glad&Ljung page 64.

Go back

2.9 What is the relation on the steady-state values among the four responses, and how is the steady-state value related to the *transfer function* when you perform a step? How is dynamic behaviour related to the poles.

Go back

3 Feedback Systems

3.1 a) The volume of water is $Ay(t)$ where A is the cross section area, so the change in volume per time unit is what mathematically?

b) Static gain, *final value theorem*...

c) The outflow caused by the pump enters in the block diagram as a *disturbance* on the input signal of the tank.

Go back

3.2 a) Apply Laplace on all terms in the definition of the controller.

b) Write down all relations you know $E(s) = R(s) - Y(s)$, $X(s) = G_V(s)U(s)$,..., and then draw.

c) Use your relations, and solve for $Y(s)$.

Go back

3.3 a)

b) Plug in $R(s)$ and $V(s)$ and apply the *final value theorem*.

c) Same as above, just a new controller.

d) What do you know about the connection between characteristic behaviour, position of poles, and the damping ratio.

Go back

3.4 **Go back**

3.5 a) The characteristic equation is

$$s(s+1)(s+3) + K(s+2) = 0$$

which gives $P(s) = s(s+1)(s+2)$ and $Q(s) = s+2$.

b) Characteristic equation:

$$s(s^2+2s+2) + K = 0$$

$P(s) = s(s^2+2s+2)$, $Q(s) = 1$.

c) Characteristic equation:

$$s(s-1)(s+6) + K(s+1) = 0$$

$P(s) = s(s-1)(s+6)$, $Q(s) = s+1$.

Go back

3.6 Derive the general closed loop *transfer function* by first deriving the *transfer function* for the inner loop.

a) Let $\alpha = 0$. The characteristic equation is then

$$s(s+2) + 4K = 0$$

Compute the poles explicitly as a function of K.

b) The characteristic equation is

$$s(s+2)+4K(1+s)=0$$

c) Characteristic equation:

$$s(s+2)+4K(1+s/3)=0$$

d) Characteristic equation:

$$s^2+2s+4+4\alpha s=0$$

Go back

3.7 a) Start by deriving the characteristic equation $(s+1)(s-1)(s+5)+K=0$ so you can define $P(s)$ and $Q(s)$

b) Characteristic equation:

$$(s+1)(s-1)(s+5)+K(1+0.5s)=0$$

Go back

3.8 Check the *root locus* to find in which order different K-values gives a stable/unstable system, more/less oscillative system.

Go back

3.9 Investigate the starting points and end points of the *root locus*.

Go back

3.10 Since $G(s)$ has no poles in the RHP, the closed loop system is stable if the Nyquist path of KG_o does not encircle -1. Note that the K will only modify the distance to the origin, not the shape of the curve.

Go back

3.11 Study the amplitude and phase of $G(i\omega)$.

Go back

3.12 a) Draw the complete Nyquist path and use the Nyquist criterion. (Note that $G_o(-i\omega)$ is the mirror image of $G_o(i\omega)$, mirrored in the real axis.)

b) Use the *final value theorem*, and that $G_o(0)$ is known from the Nyquist path.

c) Apply the Nyquist criterion to $\frac{K}{s}G(s)$

Go back

3.13 **Go back**

3.14 **Go back**

3.15 **Go back**

3.16 Check the *steady state error*, the relative damping, etc.

Go back

Go back

3.17 a) To compute the closed loop *transfer function* combine

$$\theta(s) = G(s)U(s)$$

and

$$U(s) = F(s)(\theta_{\text{ref}}(s) - \theta(s))$$

b) The control error can be computed using

$$E(s) = \frac{1}{1 + F(s)G(s)}\theta_{\text{ref}}(s)$$

To find the steady-state error, use the *final value theorem*.

c) See b).

Go back

3.18 **Go back**

3.19 **Go back**

3.20 **Go back**

3.21 **Go back**

4 Frequency Description

4.1 Start by subtracting the constant levels from all inputs and outputs. Determine the angular frequency ω of the signals using the figure. Remember $\omega = \frac{2\pi}{T}$ where T is the period time (not to be confused with any time constant of a first order system). Use the relationship saying that when $u(t) = A \sin \omega t$ the output becomes

$$y(t) = |G(\mathrm{i}\omega)|\, A \sin(\omega t + \arg G(\mathrm{i}\omega))$$

to determine gain $|G(\mathrm{i}\omega)|$ and phase shift $\arg G(\mathrm{i}\omega)$. Use that phase shift is given from time difference ΔT by $\omega \Delta T$

Go back

4.2 a) For $K = 0.5$ the open loop system is given by

$$G_{\mathrm{o}}(s) = F(s)G_{\mathrm{r}}(s)G_{\mathrm{s}}(s) = \frac{0.05(1 + s/0.02)}{s(1 + s/0.01)(1 + s/0.05)(1 + s/0.1)}$$

Use the rules in Glad&Ljung to make the Bode plot.

b) What can be said about the phase and *gain margin* when the output of the closed loop system oscillates with constant amplitude?

c) When the reference signal is $A \sin \alpha t$ the output signal becomes

$$y(t) = |G_{\mathrm{c}}(\mathrm{i}\omega)|\, A \sin(\alpha t + \arg G_{\mathrm{c}}(\mathrm{i}\omega))$$

The Bode plot of the open loop system can be used to compute $G_{\mathrm{c}}(\mathrm{i}\omega)$.

Go back

4.3 a) Check the behavior of $G(\mathrm{i}\omega)$ when $\omega \to 0$ and $\omega \to \infty$ respectively. See also if the absolute value and the *argument* decrease monotoneoulsy or not.

b) Translate the behavior of the amplitude and phase curves to a pole-zero diagram.

Go back

4.4 Check the final values of $y(t)$ against the *static gain* $G(0)$. Check also the *overshoots* of $y(t)$ against the height of the resonance peaks in $G(\mathrm{i}\omega)$. Check the frequency of the oscillation in $y(t)$ against the resonance frequency in $G(\mathrm{i}\omega)$.

Go back

4.5 Use MATLAB, in particular the command `bode` or `bodemag`.

Go back

4.6 Recall that for stable, linear systems "a sinusoid in gives a sinusoid out" after initial transients.

Go back

4.7 Recall that for stable, linear systems "a sinusoid in gives a sinusoid out" after initial transients.

Go back

4.8 **Go back**

4.9 **Go back**

4.10 Check the final values of $y(t)$ against the *static gain*. Check also the *overshoots* of $y(t)$ against the height of the resonance peaks in $G(\mathrm{i}\omega)$.

Go back

4.11 **Go back**

4.12 a)

b)

c)

d)

Go back

5 Compensation

5.1 a) Glad&Ljung gives a good description of how Bode plots can be drawn by hand.

b) A proportional controller does not affect the phase curve.

c) Try lead compensator.

5.2 Check for signs of dominating poles, pure integrations, resonance frequencies. And remember, $G_c = \frac{G_o}{1+G_o}$, so if you know the value of either one of them in a frequency, you the value of the other. This is particularily easy to use if either of them is 0, 1 or ∞. Use connections between *gain crossover frequency* and closed-loop bandwidth.

Go back

5.3 Draw asymptotic Bode plot using the guidelines in Glad&Ljung. See the discussion on lead-lag compensators in Glad&Ljung.

Go back

5.4 Start by adjusting Figure 5.4b to obtain the Bode plot of G.

Go back

5.5 a) The measures are defined in frequencies where the amplitude gain is 1 or the phase is $-180°$

b) How much can you increase the gain at ω_p without running into problems?

c) Use the *final value theorem*.

d) A *time delay* is described by the *transfer function* e^{-sT}.

e) The complex number $G(i\omega)$ is represented via $|G(i\omega)$ and $\arg G(i\omega)$

Go back

5.6 Think of all possible phase curves.

Go back

5.7 `bode` and `margin`.

Go back

5.8 **Go back**

5.9 Study $G_O(i\omega)$ and $G_C(i\omega)$ at low frequencies and around the *gain crossover frequency* of $G_O(i\omega)$, and remember that from $G_C = \frac{G_O}{1+G_O}$ you can infer the value of one if you know the value of the other

5.10

5.11 By increasing K you increase the *loop gain* and thus move the *gain crossover frequency* to the right and make the system faster. By reducing β you increase the *phase margin* and add more derivative action.

6 Sensitivity and Robustness

6.1 The *sensitivity function* is the *transfer function* from v to y.
Go back

6.2 Derive the relative model error

$$\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Make a simple plot of $G_c(i\omega)$ using the information in the problem formulation. Compare with the inverse of the relative model error.

Go back

6.3 Convert the condition that the amplitude of y is larger than the amplitude of v to the condition

$$|1 + G_o(i\omega)| < 1$$

What does this inequality say about the distance between the Nyquist curve and the origin?

Go back

6.4 Compute the *transfer function* of the closed loop system. Apply the robustness criterion using the given upper bound of the relative model error.

Go back

6.5 a) Derive the relative model order

$$\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

and plot $1/|\Delta(i\omega)|$.

b) Determine the level that $|G_c(i\omega)|$ cannot exceed.

Go back

6.6 a) The characteristic equation becomes

$$s^2(s+1) + \alpha(s^2 + s + 4) = 0$$

b) Derive the relative model error

$$\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Check where the absolute value of the inverse of the relative model error intersects $|G_c(i\omega)|$ given in the figure. It is sufficient to check the low frequency asymptote.

c) What can be said about the necessity and sufficiency of the stability conditions in a) and b)?

Go back

6.7 Check where the absolute value of the relative model error intersects $|G_c(i\omega)|$ given in the figure.
Go back

6.8 Derive the closed loop equation relating $y(t)$, $r(t)$, $v(t)$, and $n(t)$ using $Y(s) = V(s) + G_o(s)(R(s) - N(s) - Y(s))$. Then use the fact that the *sensitivity function* $S(s)$ and the *complementary sensitivity function* $T(s)$ are related as $S(s) + T(s) = 1$. (Here $T(s)$ coincides with the closed loop system.)

6.9 a) The relative model error is given by

$$\Delta(s) = \frac{G^0(s) - G(s)}{G(s)}$$

b) Use MATLAB to plot suitable Bode plots.

Go back

7 Special Controller Structures

7.1 a) Use the relationship

$$H(s) = \frac{1}{As}\left(\frac{1}{1+s/2}U(s) - V(s)\right)$$

b) Derive the *transfer function* from V to H when both feedforward and *feedback* are used.

Go back

7.2 a) Use $Y(s) = (G_{\mathrm{u}}(s)F_{\mathrm{f}}(s) + G_{\mathrm{v}}(s))\,V(s)$.

b) Recall that for stable, linear systems "a sinusoid in gives a sinusoid out" after initial transients.

Go back

7.3 Go back

8 State Space Description

8.1 Define $x_1 = \theta$, $x_2 = \dot{\theta}$, and utilize the differential equation for the motor.
Go back

8.2 For the nonlinear equation $\dot{x}_2 = f_2(x_1, x_2, u)$, the linearized equation is given by

$$\begin{aligned}\dot{x}_2 = f_2&(x_{1,0}, x_{2,0}, u_0)\\ &+ \frac{\partial f_2}{\partial x_1}(x_{1,0}, x_{2,0}, u_0) \cdot (x_1 - x_{1,0})\\ &+ \frac{\partial f_2}{\partial x_2}(x_{1,0}, x_{2,0}, u_0) \cdot (x_2 - x_{2,0})\\ &+ \frac{\partial f_2}{\partial u}(x_{1,0}, x_{2,0}, u_0) \cdot (u - u_0)\end{aligned}$$

Go back

8.3 Remember $q = \int r \rightarrow \dot{q} = r$, and $\frac{1}{s}$ denotes integration.
Go back

8.4

a) Use canonical form, or perform a partial fraction expansion first and introduce states to represent filtered inputs.

b) Use canonical form, or try to guess the states...

c) Use canonical form
Go back

8.5 Take the Laplace transform of $g(t)$.
Go back

8.6 $G(s) = C(sI - A)^{-1}B$.
Go back

8.7 b) Compare what happens to the states as $t \rightarrow \infty$, to the *transfer function* poles.

c) Check if $\det \mathcal{S}$ and $\det \mathcal{O}$ are nonzero.
Go back

8.8 The system is minimal, compute the *transfer function*.
Go back

8.9 a) For small deviations around 0, $\sin(\phi) \approx \phi$, $\cos(\phi) \approx 1$. Take $\ddot{z}$ as input.

b) $\det \mathcal{S} = \frac{1}{\alpha}(1 - \frac{1}{\alpha})^2$

Go back

8.10 **Go back**

8.11 a) A stationary point fulfills $f(x_0, u_0) = 0, \ \ y = h(x_0, u_0)$.

b) Use

$$\begin{array}{ll} A = f_x(x_0, u_0) & B = f_u(x_0, u_0) \\ C = h_x(x_0, u_0) & D = h_u(x_0, u_0) \end{array}$$

Go back

8.12 When given an input-output relation $Z(s) = \frac{B(s)}{A(s)}W(s)$, start by writing as $A(s)Z(s) = B(s)W(s)$ and go back to time-domain to see the differential equation.

Go back

8.13 a) At equilibrium, the derivative is 0 (velocity is not changing).

b)

c)

Go back

9 State Feedback

9.1 a) The closed loop system $\dot{x} = Ax + Bu$, $y = Cx$, $u = -Lx + r$ has characteristic polynomial $\det(sI - A + BL) = 0$.

b) Closed loop system is $\dot{x} = (A - BL)x + (Bl_0)r$, $y = Cx$. Use (8.15) in book with these matrices and variables. We must have $\lim_{t\to\infty} y(t) = r(t)$ when $r(t)$ is constant.

Go back

9.2 First develop the state *feedback* law, and then develop an observer to supply state estimates to replace the states in the *feedback* law. The observer should have poles faster than the designed closed loop poles.

Go back

9.3 a) Write the system in state space form by introducing three state variables corresponding to the outputs of the three left-most integrators in the figure ($\dot{z}$ = output). Design a state *feedback* controller $u = -Lx + y_{\text{ref}}$ and place the poles in -0.5.

c) Design an observer with poles to the left of the closed loop poles.

9.4 a) A *state space model* is a collection of first order derivatives...

b) Use your *state space model*! The constant l_0 is most easily found by using that $\dot{\theta} = \dot{\omega} = 0$ at steady state.

c) Introduce the integrated control error as an auxillary state.

Go back

9.5 Is the system observable?

Go back

9.6 a) Is the system *controllable*?

b) What does it mean to have a pole at $-p$?

c) Is the system observable?

Go back

9.7 a) The connection between the states and the output is given by the C matrix, and connections between the states and their derivatives are revealed by the matrix A.

Go back

9.8 **Go back**

9.9 **Go back**

9.10 **Go back**

11 Implementation

11.1 Go back

Answers

This version: 2025-08-21

1 Mathematics

1.1 a) $\frac{A}{s}$.

b) $\frac{A}{s^2}$.

c) $\frac{1}{s+2}$

d) $\frac{s}{s^2+25}$

e) $sU(s) - u(0)$

f) $sU(s)$. ($u(0) = 0$ is a common assumption in the course.)

g) $s^2U(s) - su(0) - \dot{u}(0)$

h) $s^2U(s)$. ($u(0) = \dot{u}(0) = 0$ is a common assumption in the course.)

i) A *time delayed* signal has Laplace transform, $e^{-sT}U(s)$.

Go back

1.2 a) $\lim_{t\to\infty} y(t) = 5/2$

b) $Y(s) = \frac{1}{s+2}U(s)$

c) $\lim_{t\to\infty} y(t) = 5/2$ (of course!)

Go back

1.3 a) $f(t) = 1 - e^{-t}$; 1.

b) $f(t) = -0.5e^{-t} + 0.5e^{t}$; ∞.

c) $f(t) = e^{-t} \cdot t$; 0.

Go back

1.4 a) $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, \quad t \geq 0$

b) $y(t) = 1 - 0.5e^{-t} + 0.5\sin t - 0.5\cos t$

Go back

1.5 $y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$

Go back

1.6 Roots given by $s = \pm$

Go back

1.7 a) $\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$

b) $\frac{\sqrt{2}}{10}e^{-\mathrm{i}\frac{105}{180}\pi}$

c) $1+\sqrt{3}\mathrm{i}$

d) -5

Go back

1.8

deciBel (dB_{20})	Definition		Amplification F
20	$20\log F = 20$	$\Rightarrow$	$F = 10^1 = 10$
-3	$20\log F = -3$	$\Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20\log F = 0$	$\Rightarrow$	$F = 10^0 = 1$
10	$20\log F = 10$	$\Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20\log F = -10$	$\Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

Go back

1.9 Multiplication of the two matrices gives the unit matrix.

Go back

2 Dynamic Systems

2.1 a) Differential equation

$$\ddot{\theta}(t) + \frac{1}{\tau} \cdot \dot{\theta}(t) = k_0 \cdot u$$

where

$$\frac{1}{\tau} = \frac{R_\mathrm{a} f + k_\mathrm{a} k_\mathrm{v}}{J R_\mathrm{a}} \quad k_0 = \frac{k_\mathrm{a}}{J R_\mathrm{a}}$$

b) *transfer function*

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s + 1/\tau)}$$

c) *Step response*

$$\theta(t) = k_0 \tau t - k_0 \tau^2 (1 - e^{-t/\tau})$$

Go back

2.2 (1) $K = 0.1$

(2) $K = 2.5$

(3) $K = 3$

(4) $K = 0.5$

Go back

2.3 $G(s) = \frac{10e^{-(L/V)s}}{1+3s}$

Go back

2.4 a) $a < 1$

b) $b = 2$

Go back

2.5 A–2, B–6, C–1, D–3, E–5, F-4.

2.6 G_A is slow compared to G_B, G_C is slow compared to G_D, G_E is very oscillative compared to $G_F(s)$. Consistent with dominant pole distance to origin mainly controlling speed, and angle to real line specifying oscillatory behaviour.

Go back

2.7 $\alpha > 0$ gives *overshoot*, $\alpha < 0$ gives undershoot.

Go back

2.8 a) 1.5

b) $M \approx 26\%$

c) $T_\mathrm{r} \approx 1.5$

d) $T_\mathrm{s} \approx 7.8$

Go back

2.9 G_1–C, G_3–B, G_4–A, G_5–D.

Go back

3 Feedback Systems

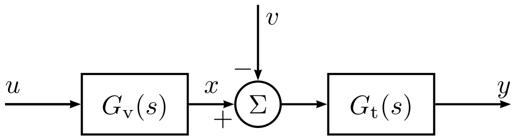


Figure 3.1a

3.1 a) transfer function of the tank is $G_t(s) = \frac{1}{s}$.

b) $k_v = 2$, $T = 5$

c) A block diagram is given in Figure 3.1a.

Go back

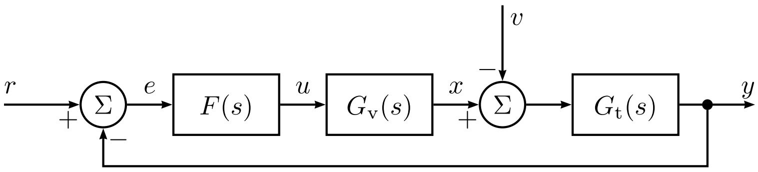


Figure 3.2a

3.2 b) A block diagram is given in Figure 3.2a.

c)

$$Y(s) = \frac{2(K_D s^2 + K_P s + K_I)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I} R(s) - \frac{s(1+5s)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I} V(s)$$

Go back

3.3 a) The poles are located at $-0.1 \pm \sqrt{0.01 - 0.4K_P}$, and are stable for all $K_P > 0$.

b) Steady state level is $5 - \frac{1}{K_P}$, but the system will be oscillatory if K_P is large.

c) As long as the system is stable, the steady state level will equal the reference.

d) The derivative part in the feedback can be used to improve the damping (move poles towards the real line) in the closed loop system, that is, to make it less oscillatory.

Go back

3.4 a) For small values of K_P the step response is slow, well damped and the steady state error is large. For increasing K_P the step response becomes faster but more oscillatory, while the error is reduced. For large K_P the amplitude of the oscillations increases, that is, the closed loop system becomes unstable.

b) The integrator in the regulator eliminates the *steady state error*. A too small value of K_I gives a large *settling time* while a too large value gives an oscillatory (finally unstable) closed loop system.

c) Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing K_D too much, however, gives that an oscillation with higher frequency appears in the *step response* and finally (approximately when $K_D > 65$) the closed loop system becomes unstable.

Go back

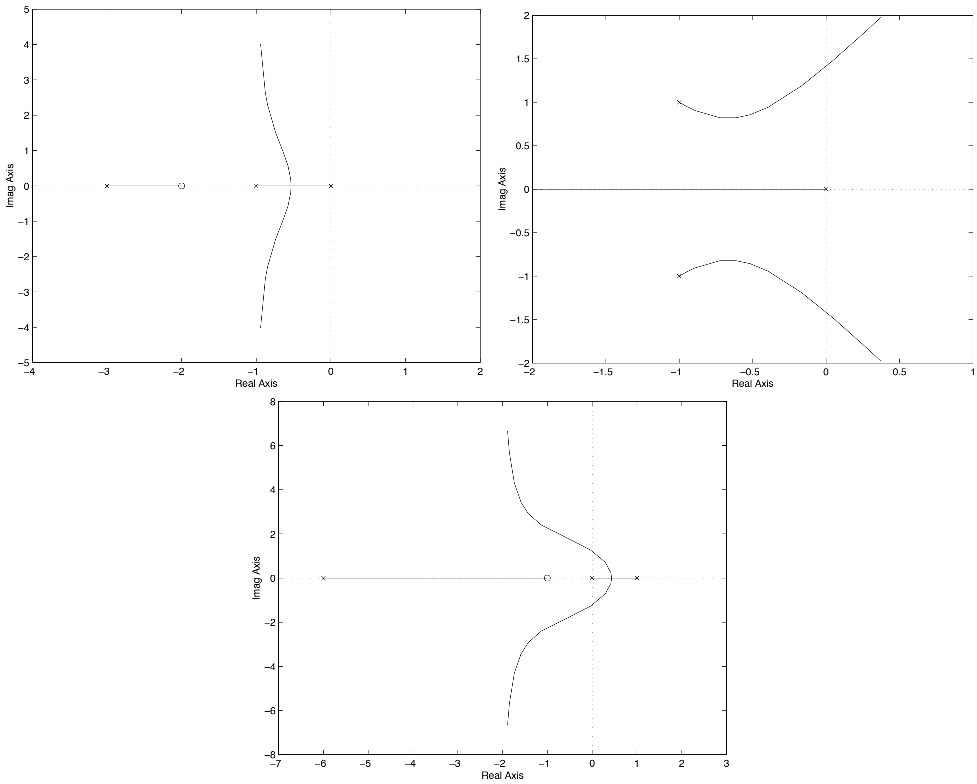


Figure 3.5a

3.5 Root loci are shown in Figure 3.5a.

b) Intersection with the imaginary axis for $K = 4$, $\omega = \pm\sqrt{2}$.

c) Intersection with the imaginary axis for $K = 7.5$, $\omega = \pm\sqrt{1.5}$.

Conclusions about the *step response* of the corresponding systems:

a) Asymptotically stable all $K > 0$.
Small K: No oscillations, larger K gives faster system.
Larger K: Oscillations. Larger K gives more oscillations.

b) Asymptotically stable for $0 < K < 4$. Oscillating all $K > 0$.
Small K: larger K gives faster system.
Larger K: larger K gives more oscillating system. Unstable for large K (> 4).

c) Asymptotically stable for $K > 7.5$. Unstable for $K < 7.5$. Stable and oscillating for $K > 7.5$. Larger K gives faster system, until the real pole becomes dominating, then larger K gives a slower system.

Go back

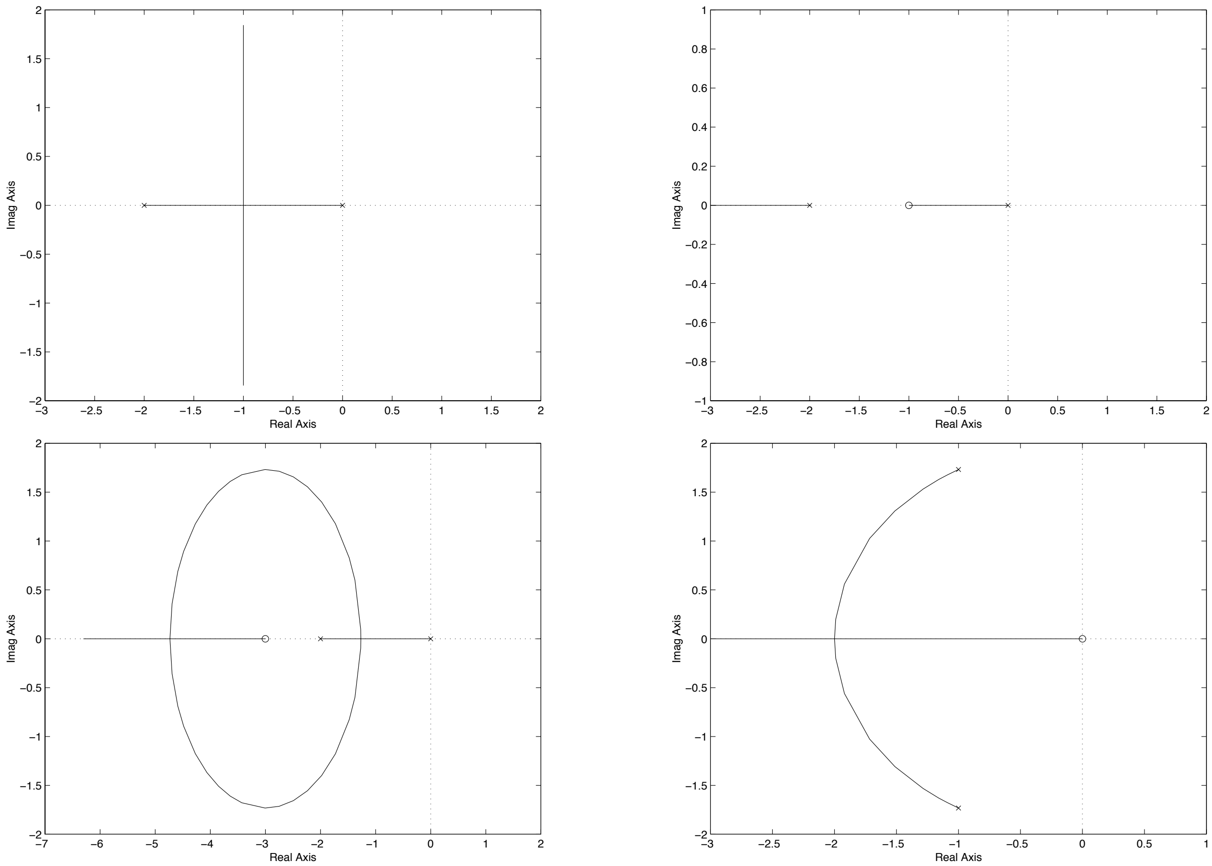


Figure 3.6a

3.6 General characteristic equation:

$$s(s+2) + 4K(1 + \alpha s) = 0$$

The root loci are shown in Figure 3.6a.

a) Asymptotically stable for all $K > 0$, oscillatory for large K.

b) Asymptotically stable for all $K > 0$, not oscillatory for any K.

c) Asymptotically stable for all $K > 0$, no oscillations for small and large K, faster for large K.

d) Asymptotically stable for all $\alpha > 0$. Oscillatory for small α. Larger α gives more damped system.

With the *tachometer feedback* we can make the system both fast and well damped. The *tachometer feedback* is equivalent to the D-part in a PID controller.

Go back

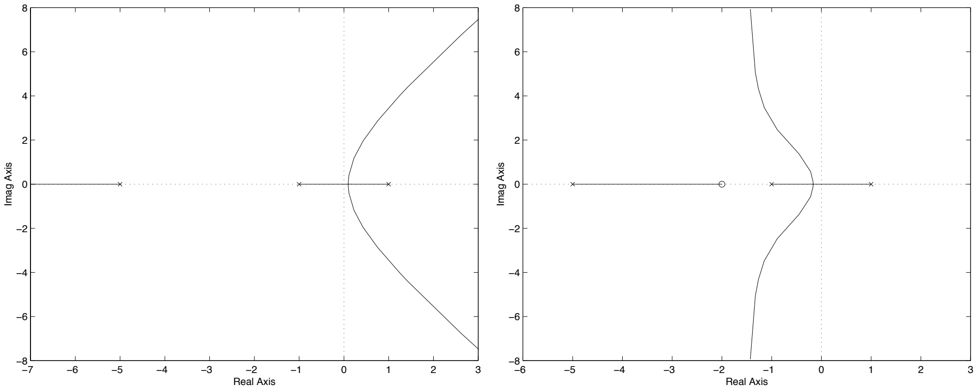


Figure 3.7a

3.7 Root loci in Figure 3.7a.

a) The system is unstable for all K.

b) Asymptotically stable for $K > 5$.

Go back

3.8

K	Step
4	C
10	D
18	B
50	A

Go back

3.9 The poles of the system all tends to points in the LHP or to $-\infty$ for large K.

Go back

3.10 a) The closed loop system is stable in (i), (ii), and (iv).

b) Stable when: (i) $K < 2.5$, (ii) $K > 0$, (iii) $K < 1/2$, and (iv) $K < 1/4$ or $K > 1/2$.

Go back

3.11 The Nyquist curves are shown in Figure 3.11a.

Go back

3.12 a) $K < 2/3$

b) $\frac{1}{1+2K}$ when $K < 2/3$

c) $K < 2/3$

Go back

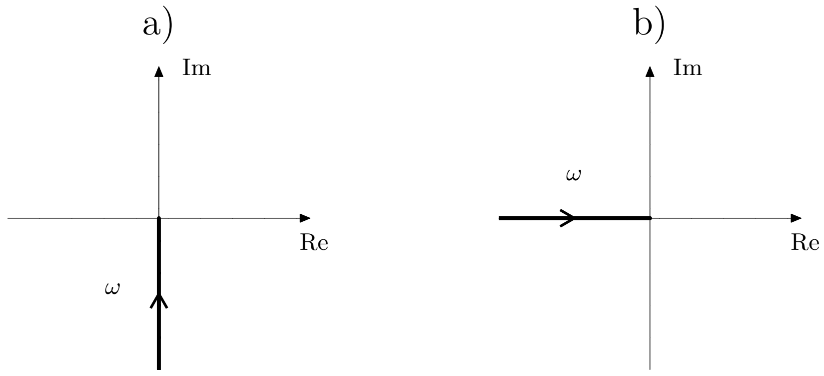


Figure 3.11a

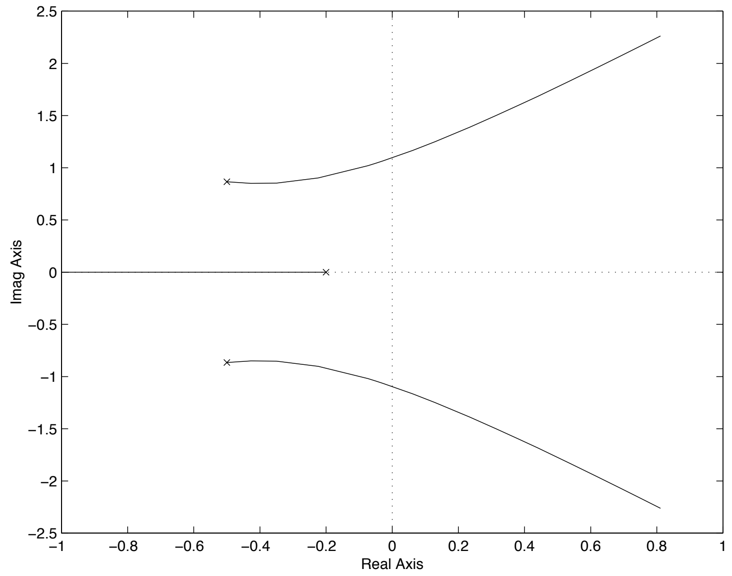


Figure 3.13a

3.13 a) The root locus with respect to K_P is shown in Figure 3.13a. When K_P increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_P \approx 6.2$, the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem 3.4. For small values of K_P the properties of the step response are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the step response increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the steady state error changes with the parameter.

b) The root locus with respect to K_I is shown in Figure 3.13b. For small K_I the response of the closed loop system is dominated by the poles on the real axis close to the origin. When K_I increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_I \approx 1.5$, the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem 3.4 a small value of K_I, that is, a pole close to the origin, gives a slow step response. When K_I increases the dominating poles become complex and the step response becomes oscillatory.

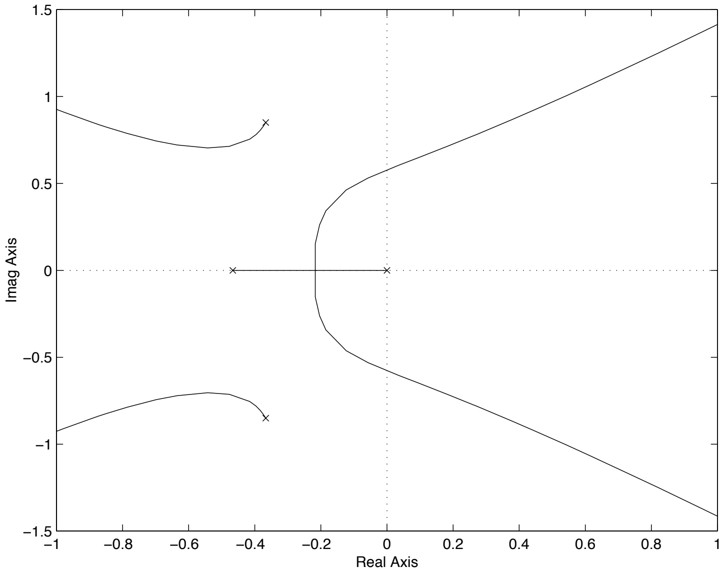


Figure 3.13b

A large settling time will typically follow if the system is slow or has poor damping. Here, the large settling time for small K_I is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

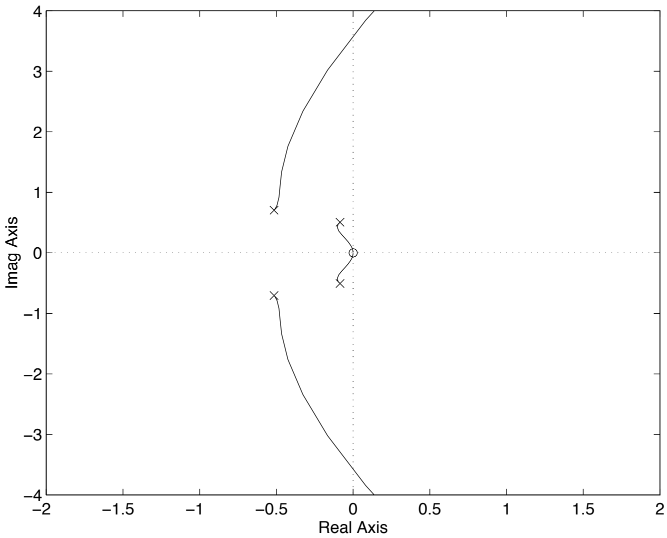


Figure 3.13c

c) The root locus with respect to K_D is shown in Figure 3.13c. When K_D increases the complex poles closest to the origin move towards the origin and and at the same time the damping of the poles is increased. When K_D increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

Go back

3.14 a) The Nyquist curve is “far away” from the point -1 for all frequencies and the step response of the closed loop system is well damped. As K_P increases the Nyquist curve grows in size and for $K_\mathrm{P} = 6.2$ the Nyquist curve reaches -1 and thus is the limit of stability.

b) For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes $|G(\mathrm{i}\omega)|$ large for low frequencies. The Nyquist curve now passes closer to -1 which results in a more oscillatory closed loop system. The system becomes unstable around $K_\mathrm{I} = 1.44$.

c) The Nyquist curve is now further away from -1 which corresponds to an improved damping of the closed loop system. The system becomes unstable around $K_\mathrm{D} = 66$.

Go back

3.15 a) $\omega_\mathrm{c} = 0.38$, $\omega_\mathrm{p} = 1.1$, $\varphi_\mathrm{m} = 94°$ and $A_\mathrm{m} = 3.1$.

b) The closed loop system is now much more oscillatory due to the reduced phase and gain margins.

c) $K_\mathrm{P} = 3.1$.

Go back

3.16 A–*iii*, B–*i*, C–*iv*, D–*ii*.

Go back

Go back

3.17 a) K_P small $\Rightarrow$ Both poles on the real axis, but one pole very close to the origin $\Rightarrow$ Slow but not oscillatory system.

$K_\mathrm{P} = 1/(4\tau^2 k_0) \Rightarrow$ Both poles in $-1/(2\tau)$, that is, faster than in (1) but still no oscillations.

K_P large $\Rightarrow$ Complex poles with large imaginary part relative to the real part, that is, oscillatory system.

b) If the reference is a step,

$$\lim_{t\to\infty} e(t) = 0$$

If the reference is a ramp,

$$\lim_{t\to\infty} e(t) = \frac{A}{K_\mathrm{P} k_0 \tau}$$

c) $\lim_{t\to\infty} e(t) = 0$

Go back

3.18 $G_\mathrm{c} = \frac{G_\mathrm{o}}{1+G_\mathrm{o}}$

Go back

3.19 a) $G_\mathrm{o} = FG$

b) $G_\mathrm{c} = \frac{FG}{1+FG}$

c) $G_{ny} = -\frac{FG}{1+FG}$

d) $G_{re} = \frac{1}{1+FG}$

Go back

3.20 a) $\frac{3A}{3+K}$

b) $F(s) = \frac{1}{s}$ (for example)

c) Poles in -2, -2. No zeros.

Go back

3.21 A–4, B–2, C–3, D–1.

Go back

4 Frequency Description

4.1 $G(s) = \frac{0.16}{s+0.16}$

Go back

4.2 a) See figure in the solution. $\omega_c = 0.025$, $\varphi_m = 31°$, $A_m = 2.5$.

b) The period time will be 108 seconds, $K = 1.25$.

c) $B = 8°$, $\beta = 0.02$ rad/s and $\varphi = -42°$.

Go back

4.3 a) Figure 4.3a in Solutions.

b) Figure 4.3b in Solutions.

Go back

4.4 A–ii, B–iii, C–iv, D–i.

Go back

4.5 a)

System	$Staticgain$	ω_B	ω_r	M_p
G_A	1	0.96		
G_B	1	3.2		
G_C	1	6.35	3.55	1.15
G_D	1	12.71	7.08	1.15
G_E	1	7.71	4.95	5.02
G_F	1	6.87	4.15	1.36

b) The bandwidth of a system is (approximately) inversely proportional to the *rise time*. The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large *overshoot*. As the damping ζ approaches 0 the resonance frequency approaches ω_0, in a second-order system $\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$

Go back

4.6 $y(t) = \frac{1}{\sqrt{5}} \sin(2t - 1/2 - 4 - \frac{\pi}{2} - \arctan 2)$.

Go back

4.7 a) $0.45 \sin(2t - 1.1)$

b) Unstable system.

c) $0.11 \sin(2t - 2.4)$

d) $0.45 \sin(2t - 2.1)$

Go back

4.8 a,b)

ω	$\|G(i\omega)\|$	$\arg G(i\omega)$
1	$1 = 0\ \text{dB}_{20}$	$-0.2\ \text{rad} = -11°$
5	$0.8 = -1.9\ \text{dB}_{20}$	$-0.9\ \text{rad} = -52°$
10	$0.5 = -6\ \text{dB}_{20}$	$-1.6\ \text{rad} = -92°$
20	$0.2 = -14\ \text{dB}_{20}$	$-2.2\ \text{rad} = -126°$

c) See Figure 4.8a in Solutions.

Go back

4.9 G_1–B, G_2–D, G_3–A, G_4–C, G_5–E.

Go back

4.10 Bode gain–step response pairs: A–iv, B–iii, C–i, D–ii.

Go back

4.11 a) B – a – 1 – II and A – b – 2 – I.

b) B – b – 1 – I and A – a – 2 – II.

Go back

4.12 a) Pole $s = -1$ and zero $s = -\alpha$.

b) The static gain is 1.

c) $1/\alpha$

d) See Figure 4.12a

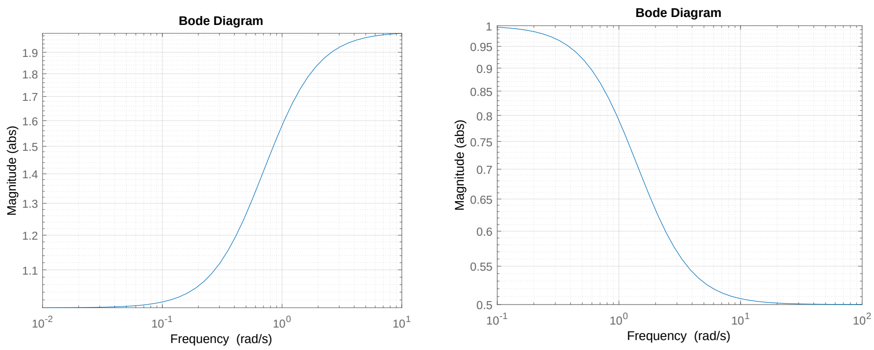


Figure 4.12a. $\mid G(i\omega) \mid$. *Left curve for* $\alpha = 0.5$. *Right curve* $\alpha = 2$.

Go back

5 Compensation

5.1 a) See Figure 5.1a.

b) Largest gain crossover frequency: 0.14 rad/s.

c) One controller that fulfills the requirements is the lead compensator (with gain adjustment)

$$F(s) = 1.9 \cdot 7\frac{s + 0.106}{s + 0.106 \cdot 7}$$

5.2 A–E–C, B–C–E, C–A–B, D–D–D, E–B–A.

Go back

5.3 One controller which satisfies the demands is

$$F(s) = 1.2 \cdot 5\frac{s + 8.0}{s + 5 \cdot 8.0} \cdot \frac{s + 1.8}{s + 1.8/84}$$

Go back

5.4 The following compensator fulfills the requirements:

$$F(s) = 4.4 \cdot \left(4\frac{s + 0.53}{s + 0.53 \cdot 4}\right)^2 \frac{s + 0.105}{s + 0.105/195}$$

Go back

5.5 a) $\omega_c = 0.8$, $\phi_m = 50°$, $\omega_p = 3$, $A_m = 10$. The closed loop is stable.

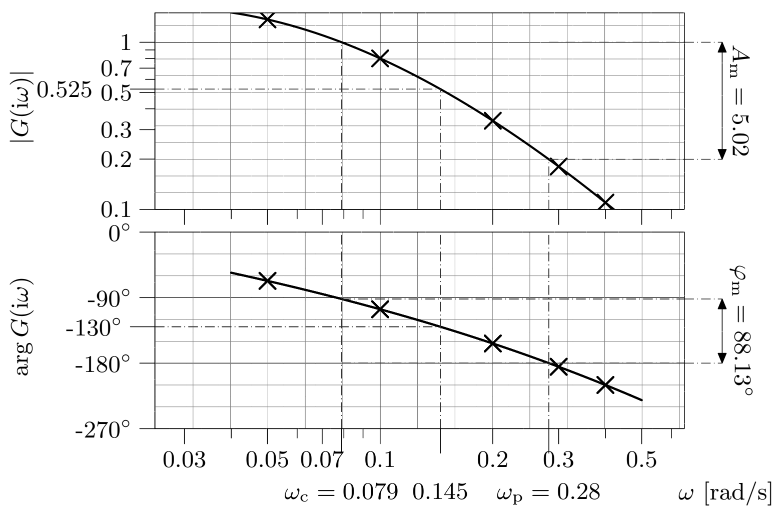


Figure 5.1a

b) $0 < K < 10$

c) $\lim_{t\to\infty} e(t) = 5$

d) $T < 0.4$

e) See Figure 5.5a.

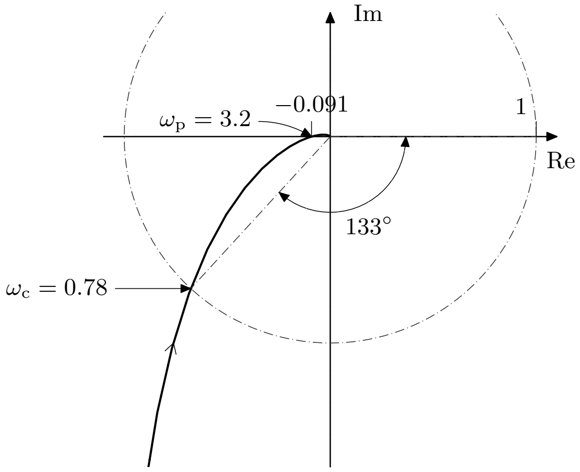


Figure 5.5a

Go back

5.6 a) Impossible to determine.

b) It is stable.

Go back

5.7 a) $\omega_c = 5$ rad/s, $\omega_p = 9.5$ rad/s, $A_m = 3.5$ and $\varphi_m = 27°$.

b)-d) See solution.

Go back

5.8 a) The maximum *gain crossover frequency* is $\omega_c = 0.27$ rad/s and it is acheived for $K = 2.86$.

b) The *steady state error* is approximately 1.75.

c)

$$F(s) = 2.39\frac{(4.4s+1)}{(0.7 \cdot 4.4s+1)}\frac{(37s+1)}{(37s+0.1)}$$

Go back

5.9 The combinations are: A - III, B - I, C - II, and D - IV.

Go back

5.10 a) One possible solution is $p = 1$, $n = 2$, and $m = 0$.

b)

$$F(s) = 3.6\frac{(0.92s+1)}{(0.92 \cdot 0.13s+1)}\frac{(3.3s+1)}{(3.3s+0.036)}.$$

5.11 A – *iii*, B – *ii*, C – *i* and D – *iv*

6 Sensitivity and Robustness

6.1 The gain of the sensitivity is:

$$|S(1\mathrm{i})| = \frac{\sqrt{2}}{\sqrt{(K-1)^2+1}}$$

and the requirement on K becomes $K > 2$.

Go back

6.2 The maximum bandwidth is $\omega_{\mathrm{B}} = 1$.

Go back

6.3 See the solution, Figure 6.3a.

Go back

6.4 Yes.

Go back

6.5 a) See the solution, Figure 6.5a.

b)

$$\left|\frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1+F(\mathrm{i}\omega)G(\mathrm{i}\omega)}\right| < \frac{1}{2}$$

Go back

6.6 a) Asymptotically stable for $\alpha > 3$. See the solution, Figure 6.6a.

b) $\alpha > 4$

c) The robustness criterion gives a sufficient but not necessary condition.

Go back

6.7 $0 \leq \gamma < \frac{1}{35}$

Go back

6.8 $y(t) = \frac{1}{\sqrt{2}}\sin(t-\frac{\pi}{4}) - \sin(t)$

Go back

6.9 a) $\frac{1}{\Delta(s)} = -\frac{s+1}{s}$

b) Stability cannot be guaranteed for $F(s) = 1$, while it can be guaranteed for the regulator from Problem 5.7.

Go back

7 Special Controller Structures

7.1 a) $F_\mathrm{f}(s) = 1$, and $h(t) = -\frac{0.1}{A\,\cdot\,2}(1 - e^{-2t})$.

b) Zero *steady state error*.

Go back

7.2 a)

$$F_\mathrm{f} = -\frac{G_\mathrm{v}}{G_\mathrm{u}} = -\frac{3(s+3)}{2(s+4)}$$

b) The amplitude of the control signal is 3.

c) $\lim_{t\to\infty} y(t) = \frac{9(1-b/2)}{12+4Kb}$

Go back

7.3 a) $F_{\mathrm{f}}(s) = -\frac{4(s+1)}{3(s+2)(s+5)}$

b) $\lim_{t\to\infty} y(t) = -0.012$

c) $\lim_{t\to\infty} y(t) = -\frac{0.012}{3K+1}$

d) $y(t)$ doesn't have a final value.

Go back

8 State Space Description

8.1

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Go back

8.2

$$\begin{aligned} \dot{x}_{1\Delta} &= x_{2\Delta} \\ \dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\ y_\Delta &= x_{1\Delta} \end{aligned}$$

Go back

8.3

$$\begin{aligned} \dot{x}_1(t) &= K_2 x_2(t) + M_l(t) \\ \dot{x}_2(t) &= -x_1(t) + x_3(t) \\ \dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t) \end{aligned}$$

Go back

8.4 a) Observable form has

$$A = \begin{pmatrix} -5 & 1 \\ -6 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}, D = 0$$

Hence,

$$\begin{aligned} \dot{x}_1(t) &= -5x_1(t) + x_2(t) + 2u(t) \\ \dot{x}_2(t) &= -6x_1(t) + 3u(t) \\ y(t) &= x_1(t) \end{aligned}$$

b) The transfer function is $\frac{0s^2+0s+6}{s^3+6s^2+11s+6}$. From this we can identify matrices in, e.g., observable form

$$A = \begin{pmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, D = 0$$

Hence,

$$\begin{aligned} \dot{x}_1(t) &= -6x_1(t) + x_2(t) \\ \dot{x}_2(t) &= -11x_1(t) + x_3(t) \\ \dot{x}_3(t) &= -6x_1(t) + 6u(t) \\ y(t) &= x_1(t) \end{aligned}$$

c) The transfer function is $\frac{4s^2+s+2}{s^3+s^2+5s+3}$ and observable form uses

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -5 & 0 & 1 \\ -3 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, D = 0$$

Hence,

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_2(t) + 4u(t) \\ \dot{x}_2(t) &= -5x_1(t) + x_3(t) + u(t) \\ \dot{x}_3(t) &= -3x_1(t) + 2u(t) \\ y(t) &= x_1(t) \end{aligned}$$

Go back

8.5

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + 2u(t) \\ \dot{x}_2(t) &= -4x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t) \end{aligned}$$

Go back

8.6 $G(s) = \frac{s}{(s+2)(s+3)}$

Go back

8.7 a) $x_1 = 1 - e^{-t}$, $x_2 = 0.5(e^{2t} - 1)$

b) No. Yes.

c) Controllable, not observable.

d) Unobservable growing state $\Rightarrow$ simulation collapses.

Go back

8.8 Poles: $1 \pm \mathrm{i}\sqrt{2}$. Zeros: -1.

Go back

8.9 a)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\alpha}x_1 - \frac{u}{\alpha} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - u \end{aligned}$$

b) $\det \mathcal{S} = \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2$. Thus, the system is *controllable* except for the case $\alpha = 1$, that is, when the two pendulums have the same lengths.

Go back

8.10 a)

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

b) $u = -5x_1 + x_2 + 3.2r$

c) $Y(s) = \frac{3.2(2s+5)}{(s+4)^2} R(s)$

8.11 b)

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2g}{x_{10}} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ -\frac{k}{mx_{10}^2} \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Go back

8.12 **Go back**

8.13 a) $x_0 = \sqrt{\frac{u_0}{c}}$. Scaling input does not scale output correspondingly.

b)

$$\Delta \dot{x}(t) = -2\frac{c}{m} x_0 \Delta x(t) + \frac{1}{m} \Delta u(t)$$

c) $-2\frac{c}{m} x_0$. The dynamics when moving away from equilibrium is faster at higher velocities.

Go back

9 State Feedback

9.1 a) State *feedback*. Poles in $\{-3, -5\}$ gives the state feedback

$$u = -6x_1 - 14x_2 + r$$

Poles in $\{-10, -15\}$ gives the state feedback

$$u = -23x_1 - 149x_2 + \tilde{r}$$

b) $G_c(s) = \frac{l_0}{s^2+8s+15}$. Use $l_0 = 15$.

Go back

9.2 State *feedback* gain $L = \begin{pmatrix} 6 & -2 \end{pmatrix}$. Observer gain $K^{\mathrm{T}} = \begin{pmatrix} 16 & 9 \end{pmatrix}$.

Go back

9.3 a)

$$\dot{x} = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u$$

b) $u = -\frac{1}{8K_1K_2}x_1 - \frac{3}{4K_1}x_2 - \frac{3}{2K_1}x_3$

c) Observer gain $K^{\mathrm{T}} = \begin{pmatrix} 6 & 12/K_2 & 8/K_2 \end{pmatrix}$

9.4 a) $x_1 = \theta$ and $x_2 = \omega$, $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} f$, $y = \begin{pmatrix} 1 & 0 \end{pmatrix}$

b) $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\mathrm{ref}}$

c) $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\mathrm{ref}} - \frac{c_2}{c_1}\hat{x}_3$

Go back

9.5 The system is observable and the poles of the observer may be placed *arbitrarily*.

Go back

9.6 a) Yes, since the system is *controllable*.

b) Closed loop poles in -3 gives

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\mathrm{ref}}$$

c) The system is observable with the sensor at x_1 or x_3. The sensor at x_1 and observer poles in -4 give $K^{\mathrm{T}} = \begin{pmatrix} 6 & 14 & 14 \end{pmatrix}$.

Go back

9.7 a) $x_2 = y$ (motor angle) and $x_1 = \dot{y}$ (angular velocity).

b) The pole locations give similar rise and *settling times*. With complex poles the maximum value of the input is lower.

c) Larger weight on the motor angle gives faster response.

d) Increasing weight on the input makes the system slower.

e) Increasing weight on the velocity makes the system slower.

Go back

9.8 a) Yes the system is *controllable*.

b) Poles in -0.1 gives the state feedback

$$u(t) = -0.13x_1(t) - 0.128x_2(t)$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large *amplification* of the measurement noise the poles of the observer should not be placed too far into the left hand plane.

d) Observer poles in -0.2 gives the observer gain

$$K = \begin{pmatrix} 0.45 \\ 0.33 \end{pmatrix}$$

Go back

9.9 a)

b)

c)

d)

Go back

9.10 a)

b)

c)

Go back

11 Implementation

11.1 $\beta_1 = 0.905$, $\alpha_1 = 19.14$, and $\alpha_2 = -18.95$.

Go back

Reglerteknik: Solutions

- **Solutions**

This version: 2025-08-21

Solutions

This version: 2025-08-21

1 Mathematics

1.1 a) A *unit step* ($A = 1$) has the Laplace transform $\frac{1}{s}$. By linearity, a *unit step* multiplied with A will have the Laplace transform $\frac{A}{s}$

a) A unit ramp ($A = 1$) has the Laplace transform $\frac{1}{s^2}$. By linearity, a unit ramp multiplied with A will have the Laplace transform $\frac{A}{s^2}$

c) $\frac{1}{s+2}$

d) $\frac{s}{s^2+25}$

e) $sU(s) - u(0)$

f) $sU(s)$. ($u(0) = 0$ is a common assumption in the course.)

g) $s^2U(s) - su(0) - \dot{u}(0)$

h) $s^2U(s)$. ($u(0) = \dot{u}(0) = 0$ is a common assumption in the course.)

i) A *time delayed* signal has Laplace transform, $e^{-sT}U(s)$.

Go back

1.2 a) Insert $\dot{y}(t) = 0$ och $u(t) = 5$ into the differential equation $\Rightarrow\ 2y(t) = 5$.

b) Apply Laplace transforms on all terms in the differential equation to arrive at $sY(s)+Y(s) = U(s)$ which we can write as $Y(s) = \frac{1}{s+2}U(s)$. We call $G(s) = \frac{1}{s+2}$ the *transfer function* of the system, and the denominator of $G(s)$ coincides with the characteristic polynomial of the differential equation.

c) With $Y(s) = \frac{1}{s+2}U(s)$ and the Laplace transform of a step, $U(s) = \frac{5}{s}$, we have $Y(s) = \frac{1}{s+2}\frac{5}{s}$. Hence $\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{1}{s+2}\frac{5}{s} = 5\lim_{s\to 0}\frac{1}{s+2} = 5\frac{1}{2}$. Note that we have derived that the amplitude of the output is the amplitude of the input (5) times the so called ***static gain*** $G(0)$ of the system.

Go back

1.3 a) Writing the function with partial fractions yields

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

The inverse transform is then computed by use of a Laplace transform table:

$$f(t) = 1 - e^{-t}$$

This means that $f(t) \to 1$ as $t \to \infty$. The same result can also be obtained by use of the *final value theorem*, that is, by computing $\lim_{s\to 0} sF(s)$.

b) Writing the function with partial fractions yields

$$F(s) = -\frac{0.5}{s+1} + \frac{0.5}{s-1}$$

The inverse transform is then computed by use of a Laplace transform table:

$$f(t) = -0.5e^{-t} + 0.5e^{t}$$

This means that $f(t)$ will grow without bound as $t \to \infty$. Here, the *final value theorem* cannot be used since $f(t)$ lacks a final value.

c) The inverse transform can be computed by use of the relation

$$\mathcal{L}^{-1}\{G(s+a)\} = e^{-at} \cdot g(t)$$

Here, $G(s) = \frac{1}{s^2}$ and $a = 1$. The inverse transform of G is $g(t) = t$, so

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} = e^{-t} \cdot t$$

which tends to 0 as $t \to \infty$. This result can also be obtained by use of the *final value theorem*.

Go back

1.4 a) Laplace of the differential equation yields

$$(s^2 + 3s + 2)Y(s) = U(s)$$

With a *unit step*, we have $U(s) = \frac{1}{s}$ leading to

$$Y(s) = \frac{1}{(s^2+3s+2)}\frac{1}{s} = \frac{1}{(s+1)(s+2)}\frac{1}{s}$$

Rewrite using partial fractions

$$Y(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

with solution

$$Y(s) = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

Laplace inverse from table 233

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Alternatively, we can use the table on page 234 directly with result (A.34) to obtain

$$y(t) = \frac{1}{2}\left(1 - \frac{2e^{-t} - e^{-2t}}{1}\right) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, \quad t \geq 0$$

b) The Laplace transform of the input

$$u(t) = 1 + \sin t$$

yields

$$U(s) = \frac{1}{s} + \frac{1}{s^2+1}$$

The differential equation

$$\dot{y}(t) + y(t) = u(t)$$

is Laplace transformed and we see the model is represented by the *transfer function* $\frac{1}{s+1}$

$$Y(s) = \frac{1}{s+1}U(s)$$

Hence, the Laplace transform of the system output is given by

$$Y(s) = \underbrace{\frac{1}{s+1}\frac{1}{s}}_{Y_1(s)} + \underbrace{\frac{1}{s+1}\frac{1}{s^2+1}}_{Y_2(s)}$$

Rewriting the first term using partial fractions leads to

$$Y_1(s) = \frac{1}{s+1}\frac{1}{s} = \frac{1}{s} - \frac{1}{s+1}$$

with inverse transform

$$y_1(t) = 1 - e^{-t}$$

Rewriting the second term using partial fractions leads to

$$Y_2(s) = \frac{1}{s+1}\frac{1}{s^2+1} = \frac{0.5}{s+1} - \frac{0.5s}{s^2+1} + \frac{0.5}{s^2+1}$$

with inverse transform

$$y_2(t) = 0.5e^{-t} - 0.5\cos t + 0.5\sin t$$

Hence, the system output is

$$y(t) = 1 - 0.5e^{-t} + 0.5\sin t - 0.5\cos t$$

Go back

1.5 The relation between inflow $z(t)$ and water level $y(t)$ is given by

$$sY(s) + Y(s) = Z(s) \Rightarrow Y(s) = \frac{1}{s+1}Z(s)$$

and the relation between control signal $u(t)$ and inflow $z(t)$ is

$$s^2Z(s) + sZ(s) + Z(s) = U(s) \Rightarrow Z(s) = \frac{1}{s^2+s+1}U(s)$$

This means that the Laplace transforms of the control signal and water level are related by

$$Y(s) = \frac{1}{(s+1)}\frac{1}{(s^2+s+1)}U(s) = \frac{1}{s^3+2s^2+2s+1}U(s)$$

which corresponds to the differential equation

$$y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$$

Go back

1.6 Solving $s^2 + as + b = 0$ yields $s = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$. Note that $\sqrt{\frac{a^2}{4} - b}$ can become both real and complex. If $a \leq 0$ we will immediately have a non-negative real part on one of the roots. If $a > 0$ we can only get a non-negative real part if $\sqrt{\frac{a^2}{4} - b}$ is real and $\sqrt{\frac{a^2}{4} - b} \geq \frac{a}{2}$ which only can occur if $b \leq 0$.

Go back

1.7 a) The absolute value is $|1 + \mathrm{i}| = \sqrt{2}$, and the *argument* is $\arctan \frac{1}{1} = \frac{\pi}{4} = 45°$. Hence, the polar form is

$$\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$$

b) The absolute value is

$$\frac{|1 + \mathrm{i}|}{5\left|1 + \sqrt{(3)}\mathrm{i}\right|} = \frac{\sqrt{2}}{5 \cdot 2} \approx 0.14$$

The *argument* is

$$\begin{aligned} \arg\left(\frac{1 + \mathrm{i}}{5\mathrm{i}(1 + \sqrt{3}\mathrm{i})}\right) &= \arg(1 + \mathrm{i}) - \arg 5\mathrm{i} - \arg(1 + \sqrt{3}\mathrm{i}) \\ &= \arctan 1 - 90° - \arctan\sqrt{3} = 45° - 90° - 60° \\ &= -105° \end{aligned}$$

Hence, the polar form is

$$\frac{\sqrt{2}}{10}e^{-\mathrm{i}\frac{105}{180}\pi}$$

c) $2e^{\mathrm{i}\frac{\pi}{3}} = 2\cos\frac{\pi}{3} + 2\mathrm{i}\sin\frac{\pi}{3} = 1 + \sqrt{3}\mathrm{i}$

d) $5e^{-\mathrm{i}\pi} = 5\cos(-\pi) + 5\mathrm{i}\sin(-\pi) = -5$

Go back

1.8 The *amplification* in decibel is computed as $10\log|F|^2 = 20\log|F|$, where F is the absolute value of the *amplification*. The *amplification* $F = 100$ hence corresponds to $20\log 100 = 40$ dB_{20}.

decibel (dB_{20})	Definition	Amplification F
20	$20\log F = 20 \quad \Rightarrow$	$F = 10^1 = 10$
-3	$20\log F = -3 \quad \Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20\log F = 0 \quad \Rightarrow$	$F = 10^0 = 1$
10	$20\log F = 10 \quad \Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20\log F = -10 \quad \Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

Note that increasing (or decreasing) the gain with a factor 10 leads to 20dB_{20} increase (decrease).

Go back

1.9 Multiplication of the two matrices gives the unit matrix.

$$\frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Go back

2 Dynamic Systems

2.1 a) We start from the equations

$$J\ddot{\theta}(t) = -f\dot{\theta}(t) + M(t) \tag{2.1}$$
$$M(t) = k_\text{a} i(t) \tag{2.2}$$
$$v(t) = k_\text{v}\dot{\theta}(t) \tag{2.3}$$

Voltage *equilibrium* gives

$$u(t) - R_\text{a} i(t) - L_\text{a}\frac{\mathrm{d}i(t)}{\mathrm{d}t} - v(t) = 0 \tag{2.4}$$

where $L_\text{a} = 0$. Equation (2.2) in (2.1) gives

$$J\ddot{\theta}(t) + f\dot{\theta}(t) = k_\text{a} i(t) \tag{2.5}$$

From (2.4) and (2.3) we get

$$i(t) = (u(t) - k_\text{v}\dot{\theta}(t))/R_\text{a}$$

which in (2.5) gives

$$J\ddot{\theta}(t) + f\dot{\theta}(t) = k_a(u(t) - k_\text{v}\dot{\theta}(t))/R_\text{a}$$

that is

$$\ddot{\theta}(t) + \frac{R_\text{a} f + k_\text{a} k_\text{v}}{J R_\text{a}}\dot{\theta}(t) = \frac{k_\text{a}}{J R_\text{a}} u(t)$$

Define some help variables to simplify expression (we define it in terms of $1/\tau$ as τ then will correspond to a time-constant of the motor later on, impossible to see right now though)

$$\frac{1}{\tau} = \frac{R_\text{a} f + k_\text{a} k_\text{v}}{J R_\text{a}} \qquad k_0 = \frac{k_\text{a}}{J R_\text{a}}$$

which gives

$$\ddot{\theta}(t) + \frac{1}{\tau}\cdot\dot{\theta}(t) = k_0 u(t) \tag{2.6}$$

b) Laplace transformation of (2.6) gives

$$(s^2 + \frac{1}{\tau}\cdot s)\theta(s) = k_0 U(s)$$

and this gives the *transfer function*

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s + 1/\tau)}$$

Note that the *transfer function* from $u(t)$ to angular velocity $\dot{\theta}(t)$ will be given by the first-order system $\frac{k_0}{(s+1/\tau)} = \frac{k_0\tau}{(s\tau+1)}$, i.e. the time-constant is τ and the *static gain* is $k_0\tau$

c) Suppose that $u(t)$ is a *unit step*, that is,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1 & t \geq 0 \end{cases}$$

that is

$$U(s) = \frac{1}{s}$$

This gives

$$\theta(s) = G(s)U(s) = \frac{k_0}{s(s+1/\tau)} \cdot \frac{1}{s} = \left(\frac{k_0\tau}{s} - \frac{k_0\tau}{s+1/\tau}\right) \cdot \frac{1}{s}$$

Inverse Laplace transformation (using either the table on page 233 on the signal $\theta(s)$ or the table on page 234 on the system $G(s)$) gives

$$\theta(t) = k_0\tau t - k_0\tau^2(1 - e^{-t/\tau})$$

that is, θ will grow unlimited when t increases (which of course comes as no surprise, if you apply a constant *voltage* on a motor, it will reach a certain angular velocity and then keep rotating.)

Go back

2.2 (1) Asymptotically stable system. Monotonic step response, that is, real poles: $K = 0.1$.

(2) Very oscillative system on the border to instability. Poles close to the imaginary axis: $K = 2.5$.

(3) Unstable system. Poles in the right half plane: $K = 3$.

(4) Asymptotically stable system. Oscillative step response, that is, complex poles in the left half plane: $K = 0.5$.

Go back

2.3 a) The inverse Laplace transform gives the step response

$$d_1(t) = \mathcal{L}^{-1}\left\{\frac{\beta}{1+sT} \cdot \frac{1}{s}\right\} = \beta(1 - e^{-t/T})$$

For the final value, we have

$$d_1(t) \to \beta, \; t \to \infty$$

This can be derived directly as β is the *static gain* of the *transfer function*

$$\lim_{t\to\infty} d_1(t) = \lim_{s\to 0} sD_1(s) = \lim_{s\to 0} s\left(\frac{\beta}{1+sT} \cdot \frac{1}{s}\right) = \beta$$

The figure gives $\beta = 10$. At the time $t = T$, the system time constant, the *step response* has reached 63% of the final value, that is,

$$d_1(T) = 0.63 \cdot 10$$

The figure gives $T = 3$, which gives the total *transfer function*

$$G(s) = \frac{10}{1+3s}$$

b) If we measure the signal $d_2(t)$ it takes $\frac{L}{V}$ time units for the material to go from the rollers down to the sensor. The total *transfer function* including this time delay then becomes

$$G(s) = \frac{10e^{-\frac{L}{V}s}}{1+3s}$$

Go back

2.4 Use the system description

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

In the first figure $\omega_0 = 1$ and $\zeta = 0.5$.

a) For the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

we have $\omega_0 = 1$ and $\zeta = 0.5a$. The *step response* is more oscillative than in the case $\zeta = 0.5$, that is, $\zeta < 0.5$. This gives $a < 1$.

b) For the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

we have $\omega_0 = b$ and $\zeta = 0.5$. The *step response* is in this case pure time scaling compared to the case $\omega_0 = 1$. The figure shows that the *step response* is twice as fast as in the case $\omega_0 = 1$. This gives $b = \omega_0 = 2$.

Go back

2.5 The pairs of plots that belong to the same system will be written in the form *pole-zero-letter–step-response-letter*. Assume we have a *unit step* for notational simplicity (it makes no difference if we scale all inputs)

Pole-zero diagram B has a single pole in the origin $G(s) = K\frac{1}{s}$ which gives a ramp as *step response* $Y(s) = K\frac{1}{s^2}$ and thus $y(t) = Kt$, that is, B–6.

Pole-zero diagram D also has a pole in the origin which gives an infinitely growing *step response* and the only possible combination left is D–3.

Pole-zero diagram A has a zero in the origin so $Y(s) = \frac{Ks}{(s+a)(s+b)}\frac{1}{s}$ which gives final value 0 if we apply the *final value theorem*, A–2.

Pole-zero diagram F has complex poles which gives an oscillative step response, F–4.

The two remaining ones are more tricky. The difference between the step responses is a slight *overshoot*, and the difference in the *transfer functions* is a zero. A system with only real poles and no zero cannot lead to any oscillation or *overshoot*, so we must have C–1, and step response E is the only alternative left for pole-zero diagram E, i.e. the addition of a zero can lead to *overshoots*.

Answer: A–2, B–6, C–1, D–3, E–5, F–4.

Go back

2.6 a) Enter the systems to check.

```
>> GA = 5/((s + 5)*(s+1))
>> GB = 25/((s + 5)*(s+5))
>> step(GA,GB);
>> pole(GA)

ans =

-5
-1

>> pole(GB)

ans =

-5
-5
```

$G_A(s)$ has a slow dominating pole (closer to the origin) which slows down the system.

We can plot the poles also

```
>> pzmap( GA, GB )
```

b) Same procedure again, but also compute absolute values and *arguments* (converted to degrees).

```
>> GC = 25/(s^2 + 5*s + 25)
>> GD = 100/(s^2 + 10*s + 100)
>> step(GC,GD);
>> pole(GC)
ans =

-2.5000 + 4.3301i
-2.5000 - 4.3301i

>> pole(GD)
ans =

-5.0000 + 8.6603i
-5.0000 - 8.6603i

>> abs(pole(GC))
ans =

5.0000
5.0000

>> abs(pole(GD))
ans =

10.0000
10.0000


>> angle(pole(GC))*180/pi
ans =

120.0000
-120.0000

>> angle(pole(GD))*180/pi

ans =

120.0000
-120.0000


>> pzmap(GC,GD)
```

Same angles and thus similiar responses in terms of *overshoot* and oscillations, but the poles of $G_C(s)$ are closer to the origin leading to a slower response.

c) and again

```
>> GE = 25/(s^2 + 1*s + 25)
>> GF = 25/(s^2 + 4*s + 25)
>> step(GE,GF);
>> pole(GE)
ans =

-0.5000 + 4.9749i
-0.5000 - 4.9749i
>> pole(GF)
ans =

-2.0000 + 4.5826i
-2.0000 - 4.5826i

>> abs(pole(GE))
ans =

5.0000
5.0000
>> abs(pole(GF))
ans =

5.0000
5.0000

>> angle(pole(GE))*180/pi
ans =

95.7392
-95.7392

>> angle(pole(GF))*180/pi
ans =

113.5782
-113.5782

>> pzmap(GE,GF)
```

Same distance to the origin, but more complex compared to real part on $G_E(s)$, thus leading to a more oscillatory response.

d) The product of the two systems is conveniently created

```
>> step(GA*GE)
```

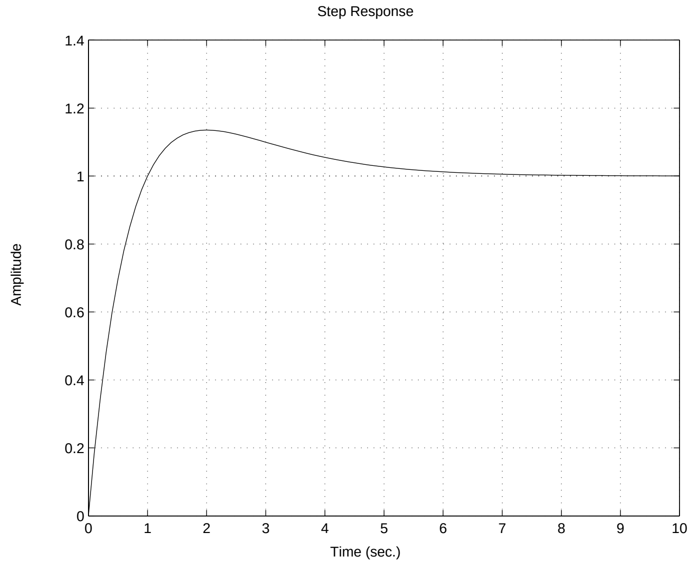
Even though there are bad complex poles in this system, the slow real pole in $G_A(s)$ is dominating and damps out the severe oscillations.

Go back

2.7 Enter the system. Here we consider the case $\alpha = 2$, that is the system has a zero in -0.5.

```
>> s = tf( 's' );
>> G1 = ( 2*s + 1 ) / ( s^2 + 2*s + 1 );
```

Plot the *step response*. `>> step( G1, 10 ); grid`



A zero located close to the origin on the negative real axis causes an *overshoot* in the *step response*. A zero on the positive real axis causes the *step response* to initially move in the negative direction. This means that in some cases the zeros of the system can have significant influence in the system properties. Systems with zeros in the right half plane normally imply extra difficulties for the design of control systems as the system initially reacts in the 'wrong' direction.

To see that we have a movement in the negative direction with a positive zero is easy to see from the equations. Let $y_1(t)$ be the step-response from the system without a zero

$$Y_1(s) = \frac{1}{s^2+2s+1}\frac{1}{s}$$

Now consider the system with a zero

$$Y(s) = \frac{\alpha s+1}{s^2+2s+1}\frac{1}{s} = \alpha s Y_1(s) + Y_1(s)$$

In time-domain this is

$$y(t) = \alpha \dot{y}_1(t) + y_1(t)$$

The output from the system will initially be proportional to the derivative of the output from the system without any zero, and if $\alpha < 0$ (positive zero) it negative since the output derivative of the system without any zero is positive.

Go back

2.8 a) The steady state value, i.e. the value the signal converges to, is 1.5.

b) The output signal almost reaches 1.9, which is slightly less than 0.4 over the final value. The *overshoot* is hence $\frac{0.4}{1.5} \approx 26\%$.

c) Find the time points where the output is 10% (0.15) and 90% (1.35) of the steady state value. The *rise time* is the difference between these values, here approximately $T_\mathrm{r} \approx 1.5$ s.

d) Find the earliest time such that the output then lies within ±5% of the steady state value. Here, the interval is $[\,1.425,\ 1.575\,]$, and the *settling time* is $T_\mathrm{s} \approx 7.8$.

Go back

2.9 G_1–C: G_1 is poorly damped, which gives an oscillatory behavior.

G_2: Can be excluded since it is the only system having *static gain* $\frac{1}{2}$, and among the *step responses* there is always more than one match for each of the present final values.

G_3–B: This case has the shortest *rise time*, and some *overshoot* due to the pair of complex poles. The *static gain* is 2.

G_4–A: The pole in -2 dominates, which gives slower *step response* than systems G_3 and G_5. The *static gain* is 1.

G_5–D: The dominating pole is in -3, which is slower than for G_3 but faster than for G_4. The *static gain* is 2.

G_6: Can be excluded due to instability.

Go back

3 Feedback Systems

3.1 a) The balance equation is, with the bottom area of the tank being 1 m^2,

$$\dot{y}(t) = x(t) - v(t)$$

that is (note that all initial conditions are zero when deriving *transfer functions*)

$$sY(s) = X(s) - V(s)$$

Hence

$$Y(s) = G_t(s)(X(s) - V(s))$$

where

$$G_t(s) = \frac{1}{s}$$

b) The *transfer function* for the *valve* is

$$G_v(s) = \frac{k_v}{1 + Ts}$$

With the input taken as a *unit step* signal, that is,

$$U(s) = \frac{1}{s}$$

it follows that

$$X(s) = \frac{k_v}{1 + Ts} \cdot \frac{1}{s}$$

The *final value theorem* gives

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = k_v$$

The *transfer function* for the *valve* corresponds to the differential equation

$$T\dot{x}(t) + x(t) = k_v u(t)$$

Assuming the intial value $x(0) = 0$ and that $u(t)$ is a step with amplitude one gives the solution

$$x(t) = k_v(1 - e^{-t/T})$$

The time constant T is the time it takes for the *step response* to reach 63% of its final value. (This comes from the observation $(1 - e^{-1}) \approx 0.63$.) From the plot it follows that $T = 5$ and $k_v = 2$, that is

$$G_v(s) = \frac{2}{1 + 5s}$$

c) The outflow caused by the pump enters in the block diagram as an additive *disturbance* on the input of the tank. The resulting structure of the open loop system is shown in Figure 3.1a.

Go back

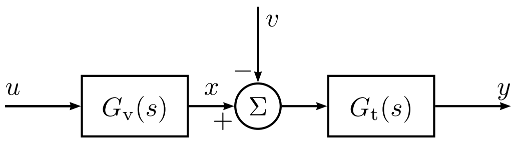


Figure 3.1a

3.2 a) Taking the Laplace transform of both sides of (3.1) gives

$$U(s) = K_P E(s) + K_I \frac{1}{s} E(s) + K_D s E(s)$$

which can be written

$$U(s) = F(s)(R(s) - Y(s))$$

if

$$F(s) = K_P + K_I \frac{1}{s} + K_D s$$

and

$$E(s) = R(s) - Y(s)$$

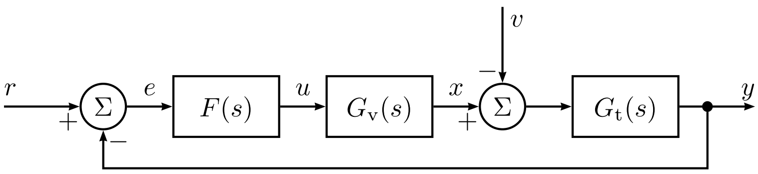


Figure 3.2a

b) By using the controller $F(s)$, the closed loop system shown in Figure 3.2a is obtained.

c) From the block diagram, the following equations are obtained:

$$\begin{aligned} Y(s) &= G_t(s)(-V(s) + F(s)G_v(s)(R(s) - Y(s))) \\ &= -G_t(s)V(s) + F(s)G_v(s)G_t(s)R(s) - F(s)G_v(s)G_t(s)Y(s) \end{aligned}$$

Collecting all terms involving $Y(s)$ on the left hand side gives

$$Y(s)(1 + F(s)G_v(s)G_t(s)) = -G_t(s)V(s) + F(s)G_v(s)G_t(s)R(s)$$

and hence

$$\begin{aligned} Y(s) = &\frac{F(s)G_v(s)G_t(s)}{1 + F(s)G_v(s)G_t(s)} R(s) \\ &- \frac{G_t(s)}{1 + F(s)G_v(s)G_t(s)} V(s) \end{aligned}$$

Inserting

$$F(s) = K_P + K_I \frac{1}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s}$$

and

$$G_t(s) = \frac{1}{s} \qquad G_v(s) = \frac{2}{1 + 5s}$$

gives

$$\begin{aligned} Y(s) = &\frac{2(K_D s^2 + K_P s + K_I)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I} R(s) \\ &- \frac{s(1 + 5s)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I} V(s) \end{aligned}$$

Note that both *transfer functions* have the same denominator, that is, stability analysis can be carried out using either of the two *transfer functions*.

Go back

3.3 (a) The needed *transfer functions* were derived in Problem 3.2 and give, putting $K_D = K_I = 0$,

$$Y(s) = \frac{2K_P}{5s^2 + s + 2K_P} R(s) - \frac{(1+5s)}{5s^2 + s + 2K_P} V(s)$$

with characteristic equation

$$5s^2 + s + 2K_P = 0$$

or, equivalently,

$$s^2 + 0.2s + 0.4K_P = 0$$

The equation has the roots

$$s = -0.1 \pm \sqrt{0.01 - 0.4K_P}$$

$K_P = 0.02$ gives the poles $\{-0.14, -0.06\}$. Since both poles are real this implies that the level will change without oscillations when a step is applied to the reference or *disturbance* signal.

$K_P = 1$ gives the poles $-0.1 \pm 0.62\mathrm{i}$. In this case the poles are complex and since the magnitude of the imaginary part is large compared to the real part, the closed loop system will not be well damped.

For $K_P \geq 0.025$ the poles are given by

$$s = -0.1 \pm \mathrm{i}\sqrt{0.4K_P - 0.01}$$

When K_P increases, the imaginary part of the poles increases while the real part is constant. The result will thus be more and more oscillatory behaviour the larger K_P is.

(b) The Laplace transforms of $r(t) = 5$ and $v(t) = 2$ are

$$R(s) = \frac{5}{s} \qquad V(s) = \frac{2}{s}$$

The Laplace transform of the level is then given by

$$Y(s) = \frac{2K_P}{(5s^2 + s + 2K_P)} \cdot \frac{5}{s} - \frac{(1+5s)}{(5s^2 + s + 2K_P)} \cdot \frac{2}{s}$$

Since the closed loop system is stable for all $K_P > 0$ the *final value theorem* can be applied. This gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 5 - \frac{1}{K_P}$$

Hence, if K_P is chosen very large the steady state level will be close to the desired level, but the system will be very oscillatory due to the location of the poles.

c) The *transfer functions* from Problem 3.3 give, putting $K_D = 0$,

$$Y(s) = \frac{2(K_P s + K_I)}{5s^3 + s^2 + 2K_P s + 2K_I} R(s) - \frac{s(1+5s)}{5s^3 + s^2 + 2K_P s + 2K_I} V(s)$$

The Laplace transform of the level, using $r(t) = 5$ and $v(t) = 2$, becomes

$$Y(s) = \frac{2(K_P s + K_I)}{(5s^3 + s^2 + 2K_P s + 2K_I)} \cdot \frac{5}{s} - \frac{s(1+5s)}{(5s^3 + s^2 + 2K_P s + 2K_I)} \cdot \frac{2}{s}$$

Provided that the coefficients K_P and K_I are chosen such that the closed loop system is stable, the steady state level can be deterimined using the *final value theorem*. This gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 5$$

which means that the desired level is reached. Hence, if the regulator coefficients are chosen such that the closed loop system is stable, the integrating part eliminates the *steady state error*.

(d) The *transfer functions* from Problem 3.2 give, putting $K_\mathrm{I} = 0$, $K_\mathrm{P} = 1$,

$$Y(s) = \frac{2(K_\mathrm{D}s + 1)}{5s^2 + (1 + 2K_\mathrm{D})s + 2} R(s) - \frac{(1 + 5s)}{5s^2 + (1 + 2K_\mathrm{D})s + 2} V(s)$$

with characteristic equation

$$5s^2 + (1 + 2K_\mathrm{D})s + 2 = 0$$

or, equivalently,

$$s^2 + (0.2 + 0.4K_\mathrm{D})s + 0.4 = 0$$

This equation can be compared to the general characteristic equation for the case of complex roots

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

where ω_0 denotes the absolute value of the roots and ζ is the relative damping. Here one gets $\omega_0 = \sqrt{0.4}$, which implies

$$\zeta = \frac{0.2 + 0.4K_\mathrm{D}}{2\sqrt{0.4}}$$

The condition $\zeta > 1/\sqrt{2}$ implies $K_\mathrm{D} > 1.7$. Poles with such damping ratio correspond to an *overshoot* of less than 5% (the *overshoot* may also be affected by the zeros). Hence, the derivative part in the *feedback* can be used to improve the damping in the closed loop system, that is, to make it less oscillatory.

Go back

3.4 a) Enter the system.

```
>> s = tf( 's' );
>> G = 0.2 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
```

Generate a proportional regulator.

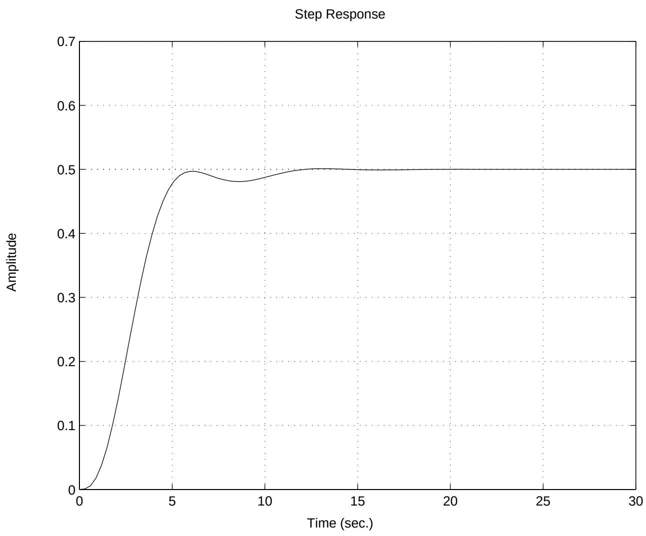
```
>> F = 1;
```

Generate the closed loop system.

```
>> Gc = feedback( F * G, 1 );
```

Compute and plot the *step response*.

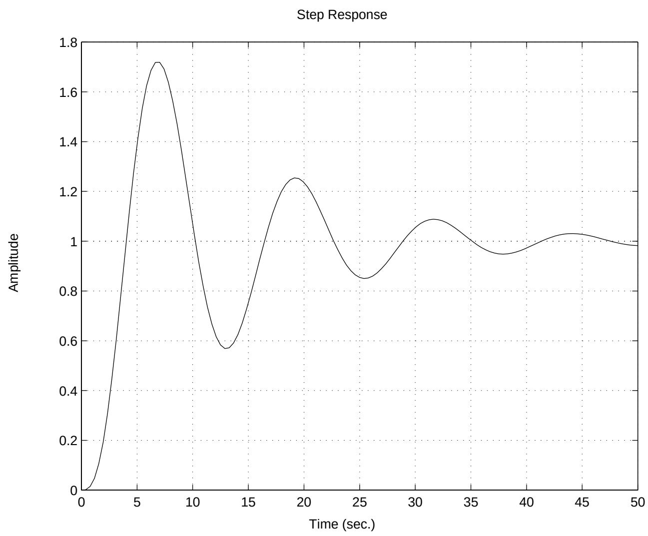
```
>> step( Gc, 30 ); grid
```



By trying some different values of K_P the following behavior can be seen: For small values of K_P the step response is slow, well damped and the steady state error is large. For increasing K_P the step response becomes faster but more oscillatory, while the error is reduced. For large K_P the amplitude of the oscillations increases over time, that is, the closed loop system becomes unstable.

b) Generate a PI controller with $K_\mathrm{P} = 1$ and $K_\mathrm{I} = 1$.
Plot the result.

```
>> KP = 1; KI = 1;
>> F = KP + KI / s;
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 ); grid
```



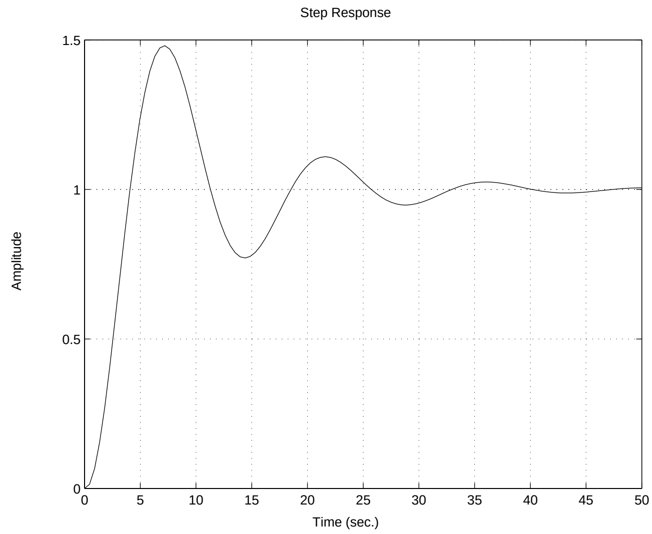
The following effects of the integrator can be found by trying some different values of K_I. (i): The integrator in the regulator eliminates the steady state error. (ii): A too small value of K_I gives a large settling time while a too large value gives an oscillatory (finally unstable) closed loop system.

c) Generate a PID controller with $K_\mathrm{P} = 1, K_\mathrm{I} = 1, K_\mathrm{D} = 2$ and $T = 0.1$.

```
>> KP = 1; KI = 1; T = 0.1; KD = 1;
>> FP = KP;
>> FI = KI / s;
>> FD = KD * s / ( s*T + 1 );
>> F = FP + FI + FD;
```

Plot the result.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 ); grid
```



Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing K_{D} too much, however, gives that an oscillation with higher frequency appears in the *step response* and finally (approximately when $K_{\mathrm{D}} > 65$) the closed loop system becomes unstable.

Go back

3.5 a) The *transfer function* for the closed loop system is

$$G_{\mathrm{c}}(s) = \frac{G_{\mathrm{o}}(s)}{1 + G_{\mathrm{o}}(s)} = \frac{K(s+2)}{s(s+1)(s+3) + K(s+2)}$$

The characteristic equation is

$$s(s+1)(s+3) + K(s+2) = P(s) + KQ(s) = 0$$

that is

$$P(s) = s(s+1)(s+3) \qquad Q(s) = s+2$$

- ◇ Starting points: ⇔ zeros of $P(s) : 0, -1, -3$
  End points: ⇔ zeros of $Q(s) : -2$
- ◇ Number of asymptotes: 2
  Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$
  Intersection with the real axis: $\frac{1}{2}[0 + (-1) + (-3) - (-2)] = -1$
- ◇ Real axis: $[-3, -2)$ and $[-1, 0]$ belongs to the *root locus*
- ◇ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation

$$\mathrm{i}\omega(\mathrm{i}\omega+1)(\mathrm{i}\omega+3) + K(\mathrm{i}\omega+2) = -\mathrm{i}\omega^3 - 4\omega^2 + (3+K)\mathrm{i}\omega + 2K = 0$$

$$\Rightarrow \left.\begin{array}{r} (-\omega^2 + 3 + K)\omega = 0 \\ -4\omega^2 + 2K = 0 \end{array}\right\} \Rightarrow \begin{array}{c} \omega = K = 0 \\ \text{(starting point)} \end{array}$$

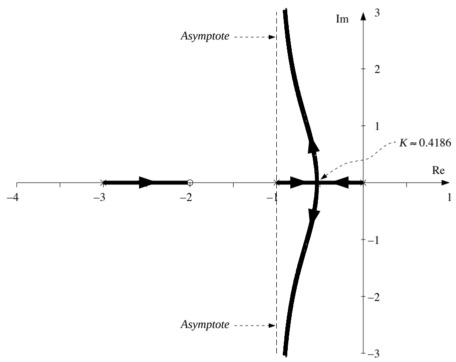


Figure 3.5a

This gives the *root locus* in Figure 3.5a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$. For small values of K there are no oscillations and the speed is increasing with increasing K. For a certain value of K the system becomes oscillating. The damping is decreasing with increasing K.

b) The *transfer function* for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{K}{s(s^2 + 2s + 2) + K}$$

The characteristic equation reads

$$s(s^2 + 2s + 2) + K = 0$$

that is

$$P(s) = s(s^2 + 2s + 2) \qquad Q(s) = 1$$

- ⋄ Starting points: $\Leftrightarrow$ zeros of $P(s) : 0, -1 \pm \mathrm{i}$
  End points: $\Leftrightarrow$ There are no zeros of $Q(s)$
- ⋄ Number of asymptotes: 3
  Directions: $\frac{1}{3}[\pi + 2k\pi] = \pi, \pm\pi/3$
  Intersection of asymptotes: $\frac{1}{3}[0 + (-1 + \mathrm{i}) + (-1 - \mathrm{i})] = -2/3$
- ⋄ Part of the real axis that belongs to the *root locus*: $(-\infty, 0]$
- ⋄ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation

$$\mathrm{i}\omega((\mathrm{i}\omega)^2 + 2\mathrm{i}\omega + 2) + K = -\mathrm{i}\omega^3 - 2\omega^2 + 2\mathrm{i}\omega + K = 0$$

$$\Rightarrow \left.\begin{array}{r} (-\omega^2 + 2)\omega = 0 \\ -2\omega^2 + K = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{ccr} \omega = K = 0 & \text{or} & \omega = \pm\sqrt{2} \\ \text{(start point)} & & K = 4 \end{array}\right\}$$

This gives the *root locus* in Figure 3.5b.

Answer: All poles are in the left half plane. That is, the system is asymptotically stable for $0 < K < 4$. The *step response* is oscillating for all K. To begin with the system will be faster with increasing K. However, for K sufficiently large the oscillating part is dominating. The damping will decrease with increasing K and for $(K \geq 4)$ the closed loop system is unstable.

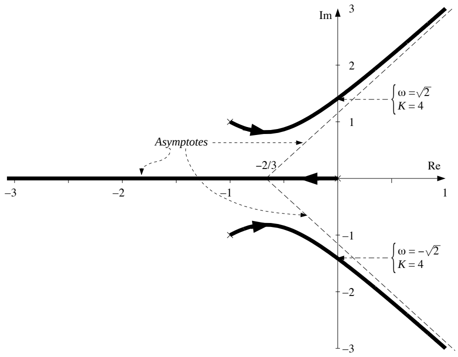


Figure 3.5b

c) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{K(s+1)}{s(s-1)(s+6) + K(s+1)}$$

The characteristic equation is

$$s(s-1)(s+6) + K(s+1) = P(s) + KQ(s) = 0$$

$$P(s) = s(s-1)(s+6) \qquad Q(s) = s+1$$

- ◇ Starting points: $\Leftrightarrow$ zeros of $P(s) : 0, 1, -6$
  End points: $\Leftrightarrow$ zeros of $Q(s) : -1$
- ◇ Number of asymptotes: $3 - 1 = 2$
  Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$
  Intersection of the asymptotes: $\frac{1}{2}[0 + 1 + (-6) - (-1)] = -2$
- ◇ Part of the real axis that belongs to the root locus: $[-6, -1)$ and $[0, 1]$
- ◇ Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation:

$$\mathrm{i}\omega(\mathrm{i}\omega - 1)(\mathrm{i}\omega + 6) + K(\mathrm{i}\omega + 1) = -\mathrm{i}\omega^3 - 5\omega^2 + (K-6)\mathrm{i}\omega + K = 0$$

$$\Rightarrow \left.\begin{array}{r} (-\omega^2 + K - 6)\omega = 0 \\ -5\omega^2 + K = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} \omega = K = 0 \\ \text{(start point)} \end{array}\right\} \text{or} \begin{array}{c} \omega = \sqrt{\frac{3}{2}} \\ K = 7.5 \end{array}$$

This gives the root locus in Figure 3.5c.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for $K > 7.5$. For small values on K the closed loop system is (as the open loop system) unstable. For $K > 7.5$ the closed loop system is stable and oscillating. As K is increasing from the critical value both the damping and the response speed are increasing (the time constant is always $\geq 1/2$s), until they both are beginning to decrease. The damping is decreasing with increasing K.

Go back

3.6 The transfer function for the closed loop system is obtained from

$$\theta(s) = \frac{1}{s}\dot{\theta}(s) = \frac{1}{s} \cdot \frac{k}{1 + s\tau} \cdot K \cdot (\theta_{\text{ref}}(s) - \alpha s\theta(s) - \theta(s))$$

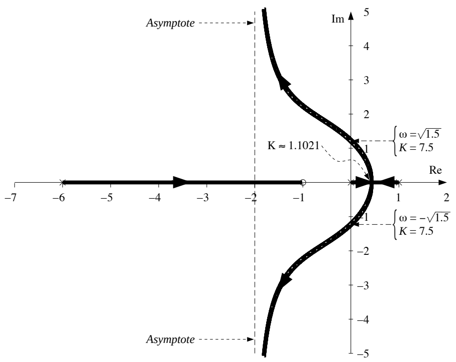


Figure 3.5c

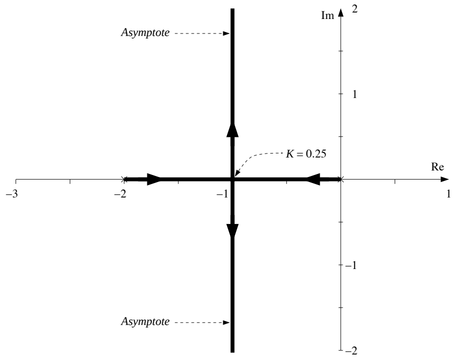


Figure 3.6a

⇒

$$G(s) = \frac{\theta(s)}{\theta_{\text{ref}}(s)} = \frac{k \cdot K}{s(1+s\tau) + k \cdot K(1+\alpha s)} = \frac{4K}{s(s+2) + 4K(1+\alpha s)}$$

The characteristic equation is:

$$s(s+2) + 4K(1+\alpha s) = 0$$

a) $\alpha = 0$. The characteristic equation is then

$$s(s+2) + 4K = s^2 + 2s + 4K = 0$$

with the solution

$$s = -1 \pm \sqrt{1-4K}$$

This gives the root locus in Figure 3.6a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

b) $\alpha = 1$. The characteristic equation is then

$$s(s+2) + 4K(1+s) = 0$$

that is

$$P(s) = s(s+2) \qquad Q(s) = 4(1+s)$$

This gives the root locus in Figure 3.6b.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

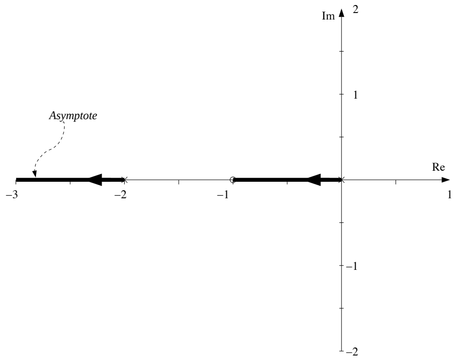


Figure 3.6b

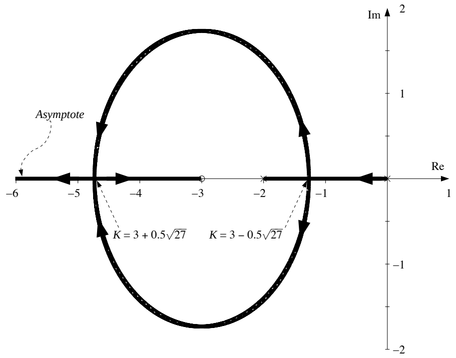


Figure 3.6c

c) $\alpha = 1/3$. The characteristic equation is then

$$s(s+2) + 4K(1+s/3) = P(s) + KQ(s) = 0$$

which gives

$$P(s) = s(s+2) \qquad Q(s) = 4(1+s/3)$$

This gives the root locus in Figure 3.6c.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

d) $K = 1$. The characteristic equation becomes

$$s(s+2) + 4(1+\alpha s) = s^2 + 2s + 4 + 4\alpha s = 0$$

that is

$$P(s) = s^2 + 2s + 4 \qquad Q(s) = 4s$$

This gives the root locus in Figure 3.6d.

Answer: All poles are in the left half plane, that is, the closed loop system is stable for all $\alpha \geq 0$. From d) it follows that the system will be more damped for larger values on α (compare b, c: in b) the system is not oscillating for any value on K). For α sufficiently large, the time constant can be arbitrary large. This is natural since the term $-\alpha\dot{\theta} \cdot K$ (D-term) that appears in the input voltage of the motor reduces the velocity of the axis. The effect is as if the motor has been drained with thick oil. With a suitable viscosity α the system can

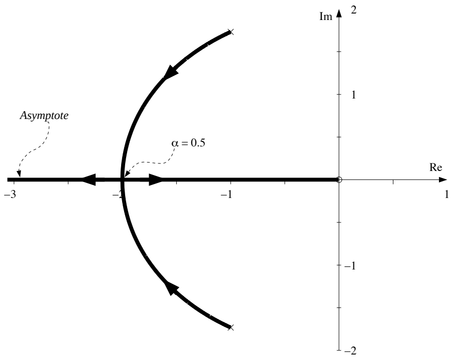


Figure 3.6d

be made fast and stable as in c). With $\alpha = 0$ as in a) and K large enough, the system is not becoming faster* just less damped.

Go back

3.7 Set

$$G(s) = \frac{1}{(s+1)(s-1)(s+5)}$$

With $U(s) = F(s)E(s)$, the *transfer function* of the closed loop system becomes

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}$$

a) Here, $F(s) = K$, so

$$G_c(s) = \frac{K}{(s+1)(s-1)(s+5) + K}$$

The characteristic equation is

$$(s+1)(s-1)(s+5) + K = 0$$

which gives

$$P(s) = (s+1)(s-1)(s+5) \qquad Q(s) = 1$$

Answer: There exists at least one pole in the RHP. Hence, the system is not asymptotically stable for any value of K.

b) Here, $F(s) = K(1 + 0.5s)$. Hence

$$G_c(s) = \frac{K(1+0.5s)}{(s+1)(s-1)(s+5) + K(1+0.5s)}$$

The characteristic equation is

$$(s+1)(s-1)(s+5) + K(1+0.5s) = 0$$

which gives

$$P(s) = (s+1)(s-1)(s+5) \qquad Q(s) = 1 + 0.5s$$

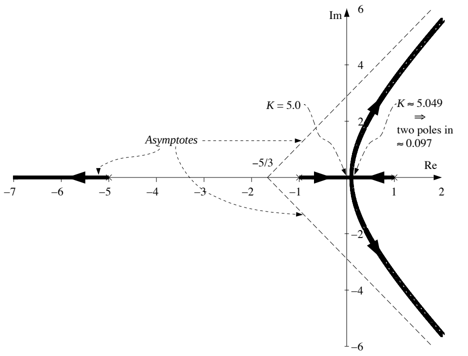


Figure 3.7a

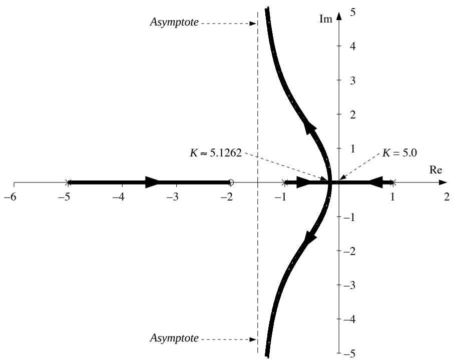


Figure 3.7b

This gives the root locus in Figure 3.7b.

Answer: The system is asymptotically stable (all poles in the LHP) if $K > 5$.

Go back

3.8 Let us move from small K to large K in our reasoning.

When K is very small all poles areal, and one is unstable as it is in the right half-plane. Consequently the magnitude of the step response grows without bound and the step response has no oscillations $\Rightarrow$ $K = 4$ corresponds to step response C.

For slightly larger K we still have poles in the right half-plane, but now a complex-conjugated pole pair, that is, the magnitude of the step response grows without bound and the step response is oscillative. $\Rightarrow$ $K = 10$ corresponds to step response D.

For even larger values of K all poles end up in the LHP. As K grows the distance to the origin of the complex-conjugated pair grows which leads to an increasing frequency on the oscillations (compare page 37 in the course book). Hence $K = 18$ corresponds to step response B and $K = 50$ to step response A.

*Note that altough the system *does* get faster with respect to risetime, this is not a very useful notion of speed in highly oscillative systems. Rather, the settling time should be used, and this property of a system is related to the real part of the poles, not their distance to the origin.

Answer:

K	Step
4	C
10	D
18	B
50	A

Go back

3.9

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \cdots + b_{n-1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{T_{n-1}(s)}{N_n(s)}$$

With a proportional *feedback* the closed loop system becomes

$$G_c(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KT_{n-1}(s)}{N_n(s) + KT_{n-1}(s)}$$

with the characteristic equation

$$N_n(s) + KT_{n-1}(s) = 0$$

that is,

$$P(s) = N_n(s) \qquad Q(s) = T_{n-1}(s)$$

- Starting points: The zeros of $N_n(s)$
  End points: The zeros of $T_{n-1}(s)$
- Number of asymptotes: 1 since $\deg N_n(s) - \deg T_{n-1}(s) = 1$
  Direction: π

When K tends to infinity, one root approaches $-\infty$, the remaining roots approaches the zeros of $T_{n-1}(s)$. The zeros of $T_{n-1}(s)$ are in the LHP according to the problem formulation. Hence, if K is large enough, the system is asymptotically stable.

Go back

3.10 The system $G(s)$ has no poles in the RHP. The closed loop system is asymptotically stable if the Nyquist curve of $KG_o(s)$ does not enclose the point -1. In the problem, Nyquist diagrams for $G(s)$ are given. The axes must hence be rescaled with a factor K.

a) (i) Yes. (ii) Yes. (iii) No. (iv) Yes.

b) (i) Stable if $0.4K < 1$, that is, $K < 2.5$.

(ii) Stable for $K > 0$.

(iii) Stable if $2K < 1$, that is, $K < 1/2$.

(iv) Stable if $4K < 1$ or $2K > 1$, that is, $K < 1/4$ or $K > 1/2$.

Go back

3.11 a) $G(i\omega) = \frac{1}{i\omega}$ gives

$$|G(i\omega)| = \frac{1}{\omega} \qquad \arg G(i\omega) = -90^\circ$$

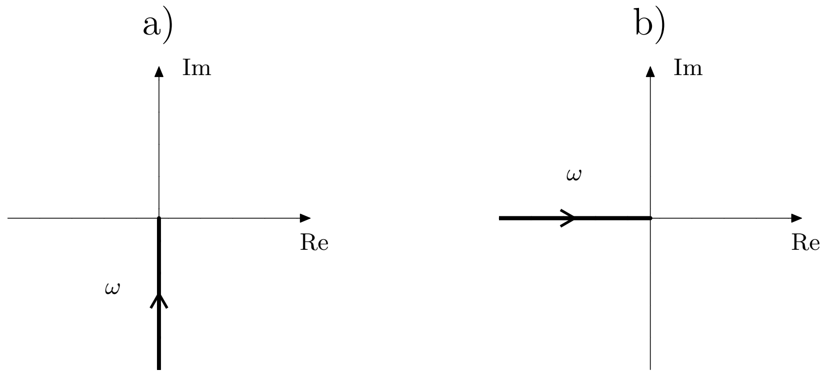


Figure 3.11a

b) $G(\mathrm{i}\omega) = \frac{1}{-\omega^2}$ gives

$$|G(\mathrm{i}\omega)| = \frac{1}{\omega^2} \qquad \arg G(\mathrm{i}\omega) = -180^\circ$$

This gives the Nyquist curves in Figure 3.11a.

Go back

3.12 a) Since $G(\mathrm{i}\omega) \to 0,\ \omega \to \infty$, we assume that the large half circle is mapped onto the origin. The small half circle is mapped onto the point 2. The point -1 must not be encircled by the curve. This means that the closed loop system is stable if $1.5 \cdot K < 1$. Hence $K < 2/3$.

b)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+KG(s)} \cdot \frac{1}{s} = \frac{1}{1+2K}$$

for $K < 2/3$ according to a.

c) The Nyquist criterion can also be applied to

$$\frac{K}{s} \cdot G(s)$$

as the open loop system. On the large half circle $\frac{1}{s} \approx 0$ which means that it is mapped onto the origin even for $\frac{1}{s} \cdot G(s)$. On the small half circle

$$s = r \cdot e^{\mathrm{i}\theta} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

we have $G(s) \approx 2$ and

$$\frac{1}{s} = \frac{1}{r}e^{-\mathrm{i}\theta}$$

Hence, it is transformed by $\frac{1}{s} \cdot G(s)$ to a large half circle in the RHP. Setting $s = \mathrm{i}\omega$ in $\frac{1}{s}$ gives the absolute value $\frac{1}{\omega}$ and the *argument* $-\pi/2$. The Nyquist curve is turned 90° and "increased" by a factor $\frac{1}{\omega}$. This gives the Nyquist path in Figure 3.12a.

Answer: The closed loop system is asymptotically stable if $\frac{3}{2}K < 1$. This means that also in this case we have $K < 2/3$.

Go back

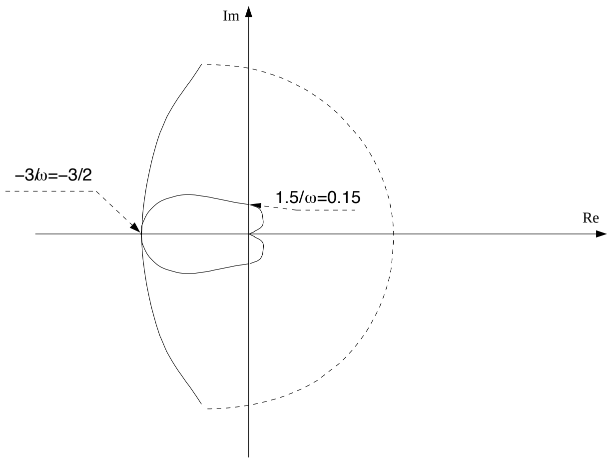


Figure 3.12a

3.13 a) The characteristic equation of the closed loop system is given by

$$(s^2 + s + 1)(s + 0.2) + K_{\mathrm{P}} \cdot 0.2 = 0$$

that is,

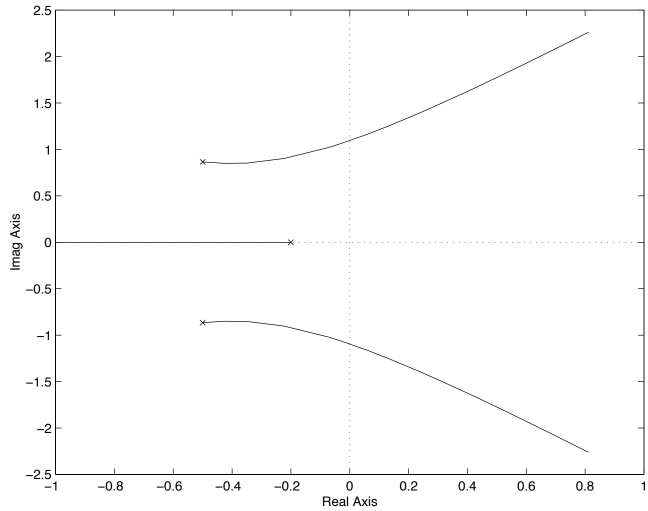
$$P(s) = (s^2 + s + 1)(s + 0.2) \qquad Q(s) = 0.2$$

Enter $P(s)$ and $Q(s)$.

```
>> s = tf( 's' );
>> P = ( s^2 + s + 1) * ( s + 0.2 );
>> Q = 0.2;
```

Draw the *root locus*. Click in the figure to determine the imaginary axis crossings.

```
>> rlocus( Q / P )
```



When K_{P} increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_{\mathrm{P}} \approx 6.2$, the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem 3.4. For small values of K_{P} the properties of the *step response* are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the *step response* increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the stationary error changes with the parameter.

b) The characteristic equation of the closed loop system using the PI controller with $K_{\rm P} = 1$ is given by

$$s((s^2 + s + 1)(s + 0.2) + 0.2) + K_{\rm I} \cdot 0.2 = 0$$

that is,

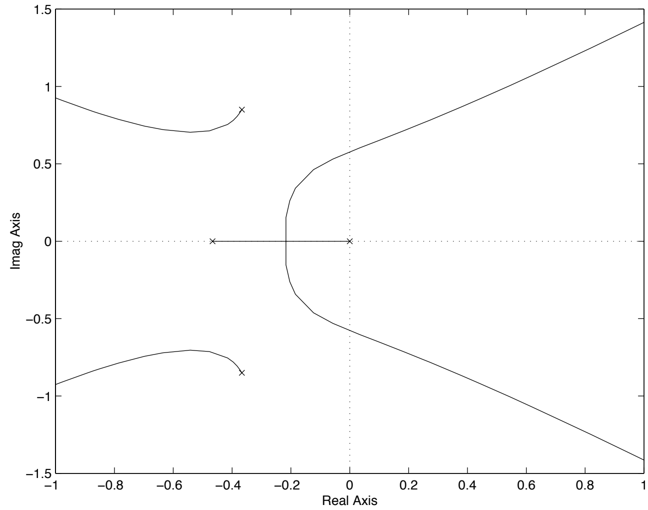
$$P(s) = s(s^3 + 1.2s^2 + 1.2s + 0.4) \qquad Q(s) = 0.2$$

Enter $P(s)$ and $Q(s)$.

```
>> P = s * ( s^3 + 1.2*s^2 + 1.2*s + 0.4 );
>> Q = 0.2;
```

Draw the root locus. Click in the figure to determine the imaginary axis crossings.

```
>> rlocus( Q / P )
```



For small $K_{\rm I}$ the response of the closed loop system is dominated by the poles on the real axis close to the origin. When $K_{\rm I}$ increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_{\rm I} \approx 1.5$, the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem 3.4 a small value of $K_{\rm I}$, that is, a pole close to the origin, gives a slow step response. When $K_{\rm I}$ increases the dominating poles become complex and the step response becomes oscillatory.

A large settling time will typically follow if the system is slow or have poor damping. Here, the large settling time for small $K_{\rm I}$ is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

c) Using PID control with $K_{\rm P} = 1, K_{\rm I} = 1$ and $T = 0.1$ the characteristic equation of the closed loop system is given by

$$(0.1s + 1)(s(s^2 + s + 1)(s + 0.2) + 0.2(s + 1)) + K_{\rm D} \cdot 0.2s^2 = 0$$

that is,

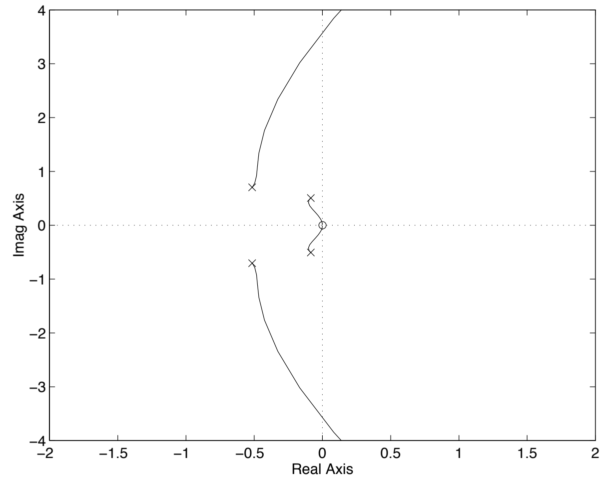
$$P(s) = (0.1s + 1)(s^4 + 1.2s^3 + 1.2s^2 + 0.4s + 0.2) \qquad Q(s) = 0.2s^2$$

Enter $P(s)$ and $Q(s)$.

```
>> P = ( 0.1*s + 1 ) * ...
       ( s^4 + 1.2*s^3 + 1.2*s^2 + 0.4*s + 0.2 );
>> Q = 0.2*s^2;
```

Draw the *root locus*. By changing the axes or using the function `zoom` the region of interest can be seen more clearly (there is also a fifth pole which is of less interest since it is located on the negative real axis, far away from the origin).

```
>> rlocus( Q / P )
>> axis([ -2 2 -4 4 ])
```

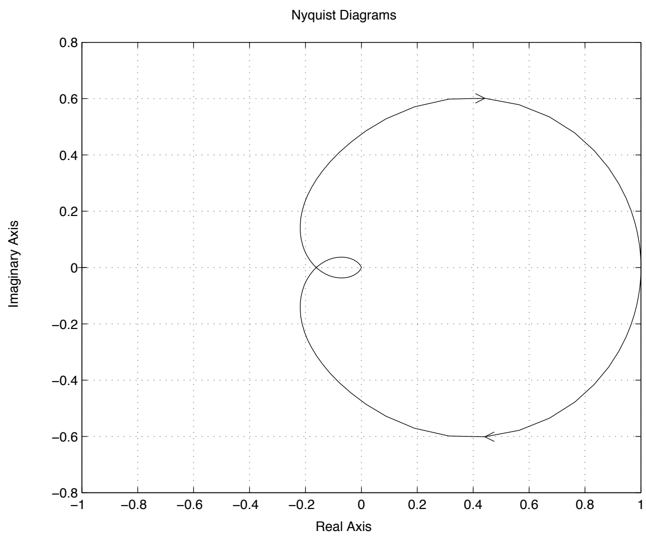


When K_D increases the complex poles closest to the origin move towards the origin and and at the same time the damping of the system is increased. When K_D increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

Go back

3.14 a) Enter the system and the regulator. Plot the Nyquist curve of the open loop system.

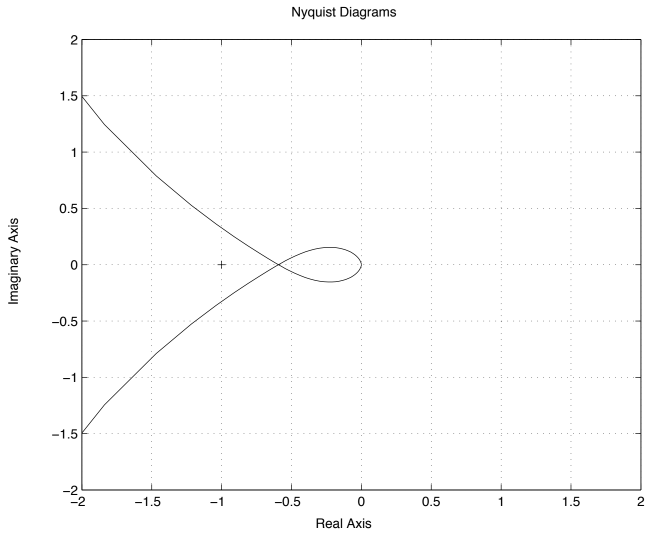
```
>> s = tf( 's' );
>> G = 0.2 / ( ( s^2 + s + 1 ) * ( s + 0.2) );
>> F = 1;
>> nyquist( F * G )
```



The Nyquist curve is “far away” from the point −1 for all frequencies and the *step response* of the closed loop system is well damped. As K_P increases the Nyquist curve grows in size and for $K_P = 6.2$ the Nyquist curve reaches −1 and thus is the limit of stability.

b) Generate a PI controller. Plot the Nyquist curve of the open loop system.

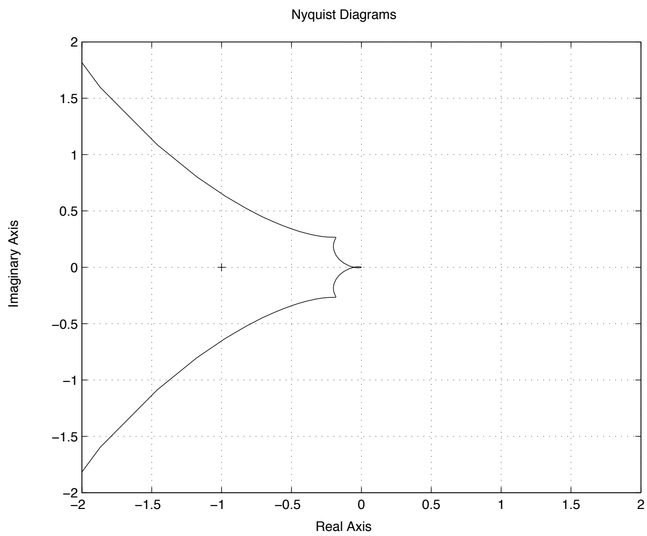
```
>> F = 1 + 1/s;
>> nyquist( F * G )
>> axis([ -2 2 -2 2 ])
```



For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes $|G(i\omega)|$ large for low frequencies. The Nyquist curve now passes closer to -1 which results in a more oscillatory closed loop system. The system becomes unstable around $K_I = 1.44$.

c) Generate a PID controller. Plot the Nyquist curve of the open loop system. Here with the parameters $K_P = 1$, $K_I = 1$, $K_D = 2$, and $T = 0.1$

```
>> F = 1 + 1/s + 2*s / ( 0.1*s + 1 );
>> nyquist( F * G )
>> axis([ -2 2 -2 2 ])
```

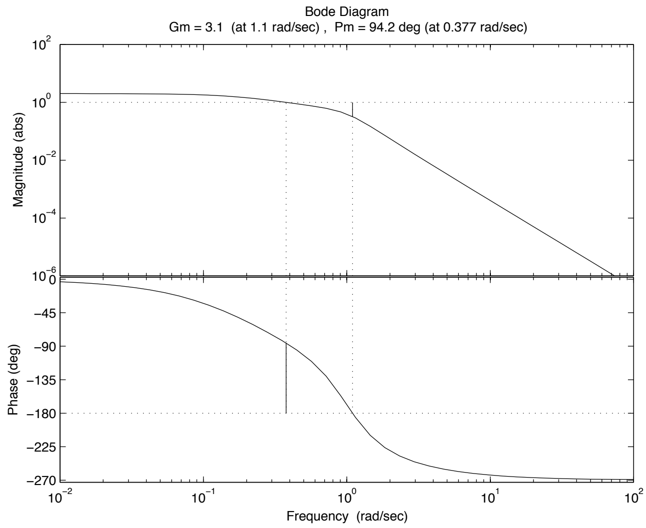


The Nyquist curve is now further away from -1 which corresponds to an improved damping of the closed loop system. The system becomes unstable around $K_D = 66$.

Go back

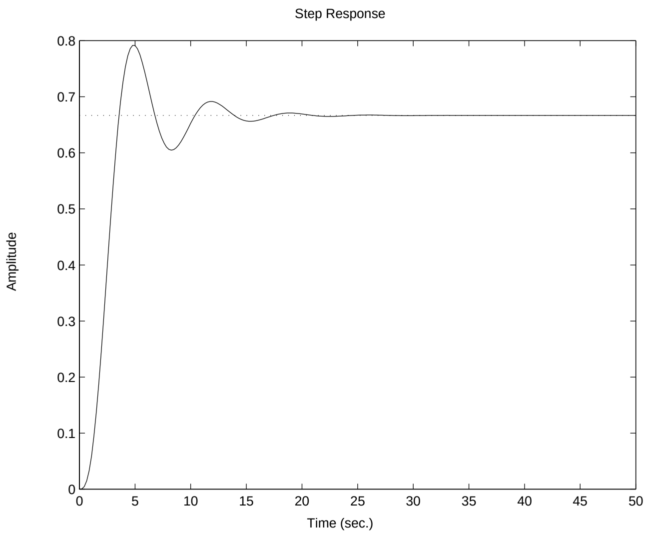
3.15 a) Enter the systems and the regulator. Make a Bode plot of the open loop system when the regulator and the system are put in series. This gives $\omega_c = 0.38$, $\omega_p = 1.1$, $\varphi_m = 94°$ and $A_m = 3.1$.

```
>> s = tf( 's' );
>> G = 0.4 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
>> F = 1;
>> margin( F * G )
```



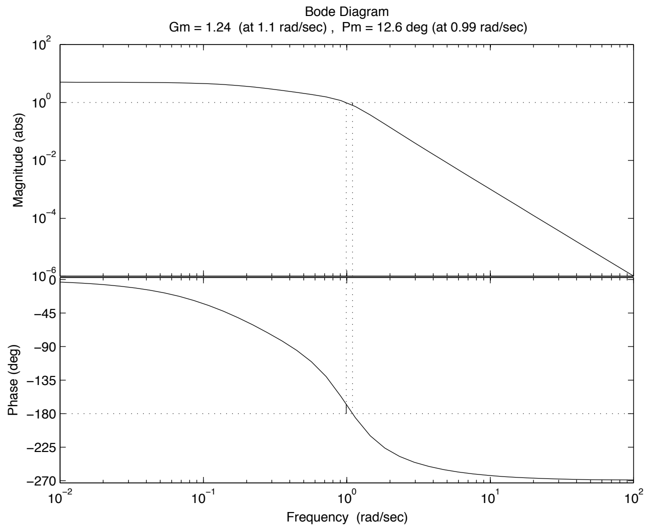
Plot the *step response*.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



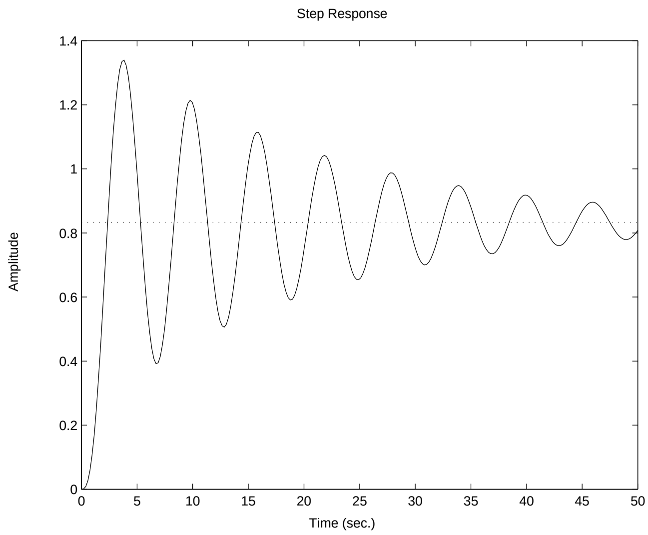
b) Increase the gain in the regulator. Make a Bode plot. The *gain crossover frequency* ω_c has increased while ω_p is the same, since only the amplitude curve is changed when the gain is changed. Both the gain and *phase margins* have decreased.

```
>> F = 2.5;
>> margin( F * G )
```



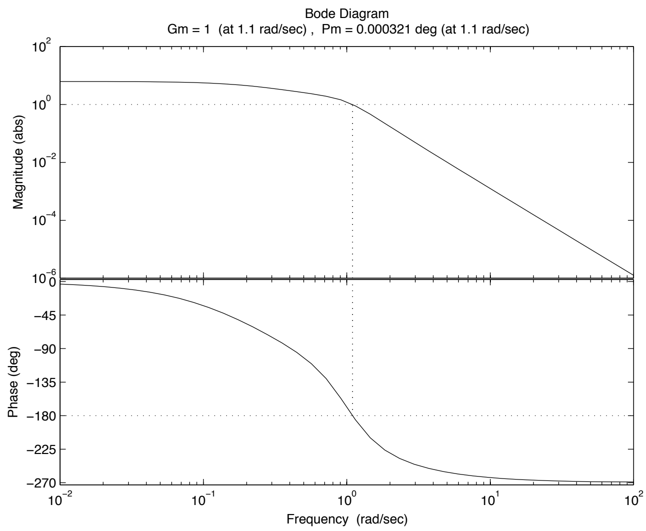
Plot the *step response*. The closed loop system is now much more oscillatory due to the reduced phase and *gain margins*.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



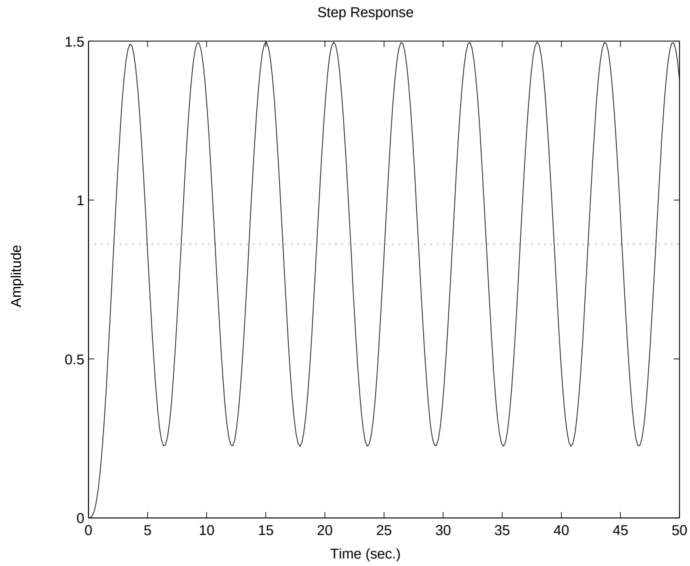
c) Increase the gain to 3.1, that is, the value of A_m in a). Both the gain and *phase margin* are at the limit between what would give an stable or unstable closed loop system. Any further increase of the gain will give an unstable closed loop system.

```
>> F = 3.1;
>> margin( F * G )
```



Plot the *step response*. The output now oscillates with constant amplitude.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



Go back

3.16 The top row gives a *steady state error* $\Rightarrow K_I = 0$. Left column less oscillative than the right one $\Rightarrow K_D \neq 0$.

Answer: A–*iii*, B–*i*, C–*iv*, D–*ii*.

Go back

3.17 a) The motor *transfer function* is (from Solution 2.1))

$$\frac{\theta(s)}{U(s)} = G(s) = \frac{k_0}{s(s+1/\tau)}$$

Feedback control

$$U(s) = F(s)(\theta_{\mathrm{ref}}(s) - \theta(s))$$

where $F(s)$ is the control law *transfer function* and θ_{ref} is the reference signal. The closed loop *transfer function* is given by

$$G_{\mathrm{c}}(s) = \frac{\theta(s)}{\theta_{\mathrm{ref}}(s)} = \frac{F(s)G(s)}{1+F(s)G(s)}$$

Proportional *feedback* $F(s) = K_{\mathrm{P}}$ and $G(s)$ according to above give

$$G_{\mathrm{c}}(s) = \frac{K_{\mathrm{P}}k_0}{s^2 + s/\tau + K_{\mathrm{P}}k_0}$$

The poles of the closed loop system are given by

$$s^2 + s/\tau + K_{\mathrm{P}}k_0 = 0$$

that is,

$$s = \frac{-1 \pm \sqrt{1 - 4\tau^2 K_{\mathrm{P}}k_0}}{2\tau}$$

(1) K_{P} small $\Rightarrow$ Both poles on the real axis, but one pole very close to the origin $\Rightarrow$ Slow but not oscillatory system.

(2) $K_{\mathrm{P}} = 1/(4\tau^2 k_0)$ $\Rightarrow$ Both poles in $-1/(2\tau)$, that is, faster than in (1) but still no oscillations.

(3) K_{P} large $\Rightarrow$ Complex poles with large imaginary part relative to the real part, that is oscillative system.

b) The *transfer function* from the reference signal to the tracking error $e = \theta_{\mathrm{ref}} - \theta$ is given by

$$E(s) = \frac{1}{1+F(s)G(s)}\theta_{\mathrm{ref}}(s) = \frac{s(s+1/\tau)}{s(s+1/\tau)+K_{\mathrm{P}}k_0}\theta_{\mathrm{ref}}(s)$$

The reference signal is a step

$$\theta_{\mathrm{ref}}(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$$

which gives

$$\theta_{\mathrm{ref}}(s) = \frac{A}{s}$$

The *final value theorem* gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot \frac{s(s+1/\tau)}{s(s+1/\tau)+K_{\mathrm{P}}k_0} \cdot \frac{A}{s} = 0$$

The reference signal is a ramp

$$\theta_{\mathrm{ref}}(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$$

which gives

$$\theta_{\mathrm{ref}}(s) = \frac{A}{s^2}$$

The *final value theorem* gives (the closed loop is asymptotically stable for all K_{P} according to a))

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot \frac{s(s+1/\tau)}{s(s+1/\tau)+K_{\mathrm{P}}k_0} \cdot \frac{A}{s^2} = \frac{A}{K_{\mathrm{P}}k_0\tau}$$

The error can be decreased by selecting K_{P} large, but according to a) the system becomes very oscillative for large K_{P}.

c) PI controller

$$u(t) = K_{\mathrm{P}}e(t) + K_{\mathrm{I}}\int_0^t e(\tau)\,\mathrm{d}\tau$$

that is

$$F(s) = K_{\mathrm{P}} + K_{\mathrm{I}}\frac{1}{s}$$

gives

$$E(s) = \frac{1}{1+F(s)G(s)}\theta_{\mathrm{ref}}(s) = \frac{s^2(s+1/\tau)}{s^2(s+1/\tau)+k_0(K_{\mathrm{P}}s+K_{\mathrm{I}})}\theta_{\mathrm{ref}}(s)$$

When θ_{ref} is a ramp according to b) we get

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

Comment: The *final value theorem* can only be used when the denominator of $G(s)U(s)$ has all zeros in the left half plane or at the origin. $G(s)$ is the system *transfer function* and $U(s)$ is the input signal.

Go back

3.18 The *transfer function* for the *loop gain* is G_{o}.

The *transfer function* from the reference signal R to the output Y is obtained by using the block diagram and observing that

$$Y = G_{\mathrm{o}}(R - Y)$$

Solving this equation for Y gives

$$Y = \frac{G_{\mathrm{o}}}{1+G_{\mathrm{o}}}R$$

that is, the *transfer function* for the closed loop system is $G_{\mathrm{c}} = \frac{G_{\mathrm{o}}}{1+G_{\mathrm{o}}}$.

Go back

3.19 a) The *loop gain*, G_{o}, is FG.

b) The influence of the *disturbance* ($N = 0$) can be neglected. Use the solution to problem Solution 3.18. The *transfer function* from R to Y is $G_{\mathrm{c}} = \frac{FG}{1+FG}$, that is, $Y = G_{\mathrm{c}}R$.

c) The influence of the reference signal can be neglected. ($R = 0$). The block diagram gives

$$Y = FGE = -FG(Y+N)$$

which implies that the *transfer function* from N to Y is $G_{ny} = -\frac{FG}{1+FG}$.

d) The influence of the *disturbance* can be neglected ($N = 0$). The block diagram gives

$$E = R - Y = R - FGE$$

Solving for E gives

$$E = \frac{1}{1 + FG} R$$

that is, the *transfer function* from R to E is $G_{re} = \frac{1}{1+FG}$.

Go back

3.20 a) The *transfer function* from reference signal to error signal is (see Solution 3.19d)

$$E = \frac{1}{1 + FG} R(s) = \frac{1}{1 + \frac{K}{(s+1)(s+3)}} R(s) = \frac{(s+1)(s+3)}{(s+1)(s+3) + K} R(s)$$

$r(t)$ step $\Rightarrow$ $R(s) = \frac{A}{s}$. The steady state value of the error is given by the *final value theorem*

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{(s+1)(s+3)}{(s+1)(s+3) + K} A = \frac{3A}{3 + K}$$

b) In order to make the *steady state error* equal to zero the regulator has to contain an integrator. Using, for example, $F(s) = \frac{1}{s}$ one gets

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{A}{1 + F(s)G(s)} = \lim_{s\to 0} \frac{A}{1 + \frac{1}{s}\frac{1}{(s+1)(s+3)}} = 0$$

Notice though, that the integrating *feedback* normally has to be combined with proportional *feedback*.

c) The *transfer function* from R to Y using $F(s) = 1$ is

$$G_c(s) = \frac{FG}{1 + FG} = \frac{1}{(s+1)(s+3) + 1} = \frac{1}{s^2 + 4s + 4} = \frac{1}{(s+2)^2}$$

The system has two poles in -2 and no zeros.

Go back

3.21
- The four *step responses* are characterized by, for example, that A and D have a *steady state error*, while C and B do not. Further, A shows better damping than D, and C shows better damping than B. It can also be noticed (although it is not as apparent as the other characteristics) that the error decays more slowly in C than in B.
- The four regulators are characterized by, for example, that regulators 1 and 4 don't have any integral action. Regulator 2 has more integral action than 3, and regulator 4 gives better damping than 1.
- The derivative part in the regulator improves the damping, while integral action eliminates the *steady state error* and reduces the damping. Besides, for small values of K_I, the error will decay slowly to zero.

Answer: A–4, B–2, C–3, D–1.

Go back

4 Frequency Description

4.1 If we let $\bar{u}(t)$ and $\bar{y}(t)$ denote the actual temperature and the measured temperature, respectively, we can divide the temperatures into their mean values and variations as follows:

$$\bar{u}(t) = u_0 + u(t)$$

and

$$\bar{y}(t) = y_0 + y(t)$$

where $u_0 = y_0 = 30$ °C.

The thermometer is modeled as a first order linear system

$$\frac{Y(s)}{U(s)} = G(s) = \frac{a}{s+b}$$

Since

$$u(t) = A\sin(\omega t)$$

it follows that after the transients have vanished (that is, in steady state)

$$y(t) = |G(\mathrm{i}\omega)|\, A\sin(\omega t + \phi)$$

where (assuming a positive)

$$\phi = \arg(G(\mathrm{i}\omega)) = \arg(a) - \arg(i\omega + b) = 0 - \arctan(\omega/b)$$

From the relationship $\omega = 2\pi/T$ and from the figure the following is obtained:

1. $\omega = \frac{2\pi}{0.314\,\cdot\,60}$ rad/s $= 0.33$ rad/s
2. $\phi = \frac{-0.056}{0.314} \cdot 2\pi$ rad $= -1.12$ rad
3. $|G(\mathrm{i}\omega)| = \frac{0.9}{2.0} = 0.45$

Hence

$$\tan(\phi) = -\frac{\omega}{b} \quad \Rightarrow \quad b = \frac{0.33}{2.066} = 0.16$$

and

$$|G(\mathrm{i}\omega)| = \frac{a}{\sqrt{\omega^2 + b^2}} \quad \Rightarrow \quad a = 0.16$$

Answer:

$$G(s) = \frac{0.16}{s + 0.16}$$

Go back

4.2 The equation

$$\omega = \dot{\psi}$$

and

$$T_1 \cdot \dot{\omega} = -\omega + K_1 \cdot \delta$$

give the *transfer function*

$$G_s(s) = \frac{K_1}{s(1+T_1 s)} = \frac{0.1}{s(1+s/0.01)}$$

The *transfer function* of the rudder machine is

$$G_r(s) = \frac{1}{1+sT_2} = \frac{1}{1+s/0.1}$$

and the controller has the *transfer function*

$$F(s) = K\frac{1+s/a}{1+s/b} = K\frac{1+s/0.02}{1+s/0.05}$$

a) $K = 0.5$ gives

$$G_o(s) = F(s)G_r(s)G_s(s) = \frac{0.05(1+s/0.02)}{s(1+s/0.01)(1+s/0.05)(1+s/0.1)}$$

It thus follows that

$$|G_o(i\omega)| = \frac{0.05\sqrt{1+(\frac{\omega}{0.02})^2}}{\omega\sqrt{1+(\frac{\omega}{0.01})^2}\sqrt{1+(\frac{\omega}{0.05})^2}\sqrt{1+(\frac{\omega}{0.1})^2}}$$

with low frequency asymptote

$$|G_o(i\omega)| \to \frac{0.05}{\omega},\ \omega \to 0$$

and

$$\arg G_o(i\omega) = \arctan\frac{\omega}{0.02} - 90^\circ - \arctan\frac{\omega}{0.01} - \arctan\frac{\omega}{0.05} - \arctan\frac{\omega}{0.1}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $\left|\frac{0.05}{0.005}\right| = 10$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		0.01		0.02		0.05		0.1	
Slope	-1		-2		-1		-2		-3

The phase shift is drawn based on a couple of samples:

Frequency [rad/s]	0.005	0.01	0.02	0.04	0.08
Phase	-111°	-125°	-142°	-163°	-194°

The Bode plot in Figure 4.2a gives: $\omega_c = 0.026$ rad/s, $\varphi_m = 32^\circ$, $A_m = 4.2$.

b) The system starts to oscillate if K is chosen so that $\arg(G_o(i\omega_c)) = -180^\circ$. This gives the *gain crossover frequency* $\omega_c = \omega_p = 0.06$ rad/s. This implies that the gain should be amplified 4.2 times. Therefore, choose $K = 0.5 \cdot 4.2 = 2.1$.

$$\omega = 2\pi/T \quad \Rightarrow \quad T = \frac{2\pi}{\omega_c} = \frac{2\pi}{0.06} = 105 \text{ s}$$

Answer: The period time will be 105 seconds, and $K = 2.1$.

c)

$$\Psi_{ref}(t) = A\sin(\alpha t)$$

gives

$$\Psi(t) = B\sin(\beta t + \varphi)$$

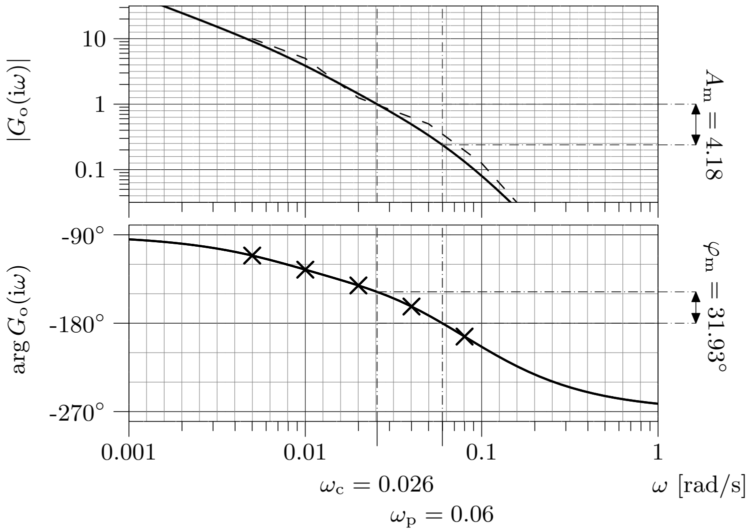


Figure 4.2a

where $A = 5°$, $\alpha = 0.02$, $\beta = \alpha$, $B = A\,|G_c(i\alpha)|$ and $\varphi = \arg G_c(i\alpha)$. The *transfer function* for the closed loop system when $K = 0.5$ is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)}$$

where

$$|G_o(i0.02)| = 1.44 \qquad \arg G_o(i0.02) = -142°$$

That is

$$G_o(i0.02) = -1.135 - i0.886$$

which gives

$$|G_c(i0.02)| = \frac{1.44}{\sqrt{0.135^2 + 0.886^2}} = 1.61 \quad \Rightarrow \quad B = 8°$$

and

$$\arg G_c(i0.02) = -142° + 180° - \arctan(\frac{0.886}{0.135}) = -0.76 \text{ rad}$$

Answer: $B = 8°$, $\beta = 0.02$ rad/s and $\varphi = -0.76$ rad.

Go back

4.3 a) As $\omega \to 0$, $|G(i\omega)| \to \infty$ and $\arg G(i\omega) \to -90°$. The gain is first decreasing (low frequencies). It then increases, and finally decreases again (approaching zero for high frequencies). The phase shift is increasing at low frequencies. As the frequency becomes higher the phase shift is positive in an interval until it decreases towards $-90°$. This gives the plot in Figure 4.3a.

b) A system with a Bode plot as the one shown above must have one pole in the origin since $\arg G(i\omega) \to -90°$ as $\omega \to 0$. Then two break points appear (up), since there is a positive phase shift. After that, there must be two break points (down), since the phase shift should approach $-90°$. Hence, the plot in Figure 4.3b is possible.

Go back

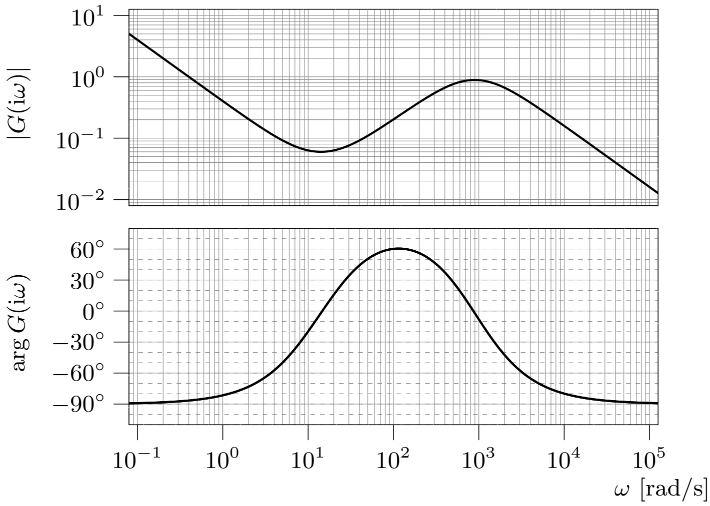


Figure 4.3a

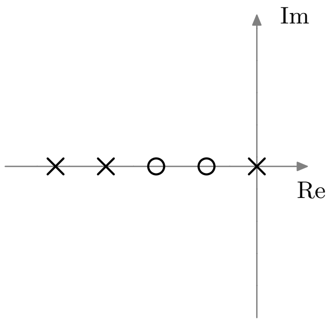


Figure 4.3b. Pole-zero diagram. Not accurate in scale; the diagram shall only be interpreted as a right to left ordering of poles and zeros, with the first pole at the origin.

4.4 From the final value of step response B (the only one greater than 1) and static gain in Bode gain iii (the only one greater than 1), the step response–Bode gain pair B–iii follows since an input of amplitude of 1 should lead to an output larger than 1 if the static gain is larger than 1. Step responses C and A have approximately the same overshoots but different fundamental frequencies. Bode gains ii and iv have equal resonance peaks but iv has a lower resonance frequency. This gives the combinations C–iv and A–ii. The remaining combination is D–i, which is a good match with small overshoot (resonance peak), and final value 1 consistent with (static gain) 1.

Go back

4.5 a) Enter the system and make a Bode plot.

```
>> s = tf( 's' );
>> GA=5/(s^2+6*s+5)
>> bode( GA )
```

Sometimes you might want to increase resolution and use specific frequencies, and only plot the amplitude

1000 points from 0.1 to 10

```
>> w = logspace(-1,1,1000);
>> bodemag( GA , w)
```

Use, for example, curve handles and "Characteristics" in the right click menu to find static gain, bandwidth, resonance frequency, and resonance peak. The other systems are treated in the same way. The results can be summarized in the following table. (Note that gain values may be presented in dB_{20} in MATLAB.)

System	$G(0)$	ω_B	ω_r	M_p
G_A	1	0.96		
G_B	1	3.2		
G_C	1	6.35	3.55	1.15
G_D	1	12.71	7.08	1.15
G_E	1	7.71	4.95	5.02
G_F	1	6.87	4.15	1.36

b) Using the results in a) and in Problem 2.6, the following observations can be made. (i): The bandwidth of a system is (approximately) inversely proportional to the *rise time*. High bandwidth implies a short *rise time* and hence a fast system. (ii): The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large *overshoot*.

Go back

4.6 From the frequency responce interpretation of the *transfer function* ("a sinusoid in gives a sinusoid out") and the input being

$$u(t) = 2\sin(2t - 1/2)$$

it follows that the output is

$$y(t) = 2\,|G(\mathrm{i}2)|\sin(2t - 1/2 + \arg G(\mathrm{i}2))$$

Here $G(s) = \frac{e^{-2s}}{s(s+1)}$, and hence

$$|G(\mathrm{i}2)| = \frac{1}{2\sqrt{2^2+1}} = \frac{1}{2\sqrt{5}}$$
$$\arg G(\mathrm{i}2) = -4 - \frac{\pi}{2} - \arctan 2$$

Go back

4.7 The input is a sinusoid with amplitude 1 and angular frequency $\omega = 2$ rad/s.

a) $0.45\sin(2t - 1.1)$.
(Gain: $\left|\frac{1}{\mathrm{i}2+1}\right| = \frac{1}{\sqrt{5}} \approx 0.45$, phase: $-\arg(\mathrm{i}2+1) \approx -1.1$ rad $= -63°$.)

b) The system is unstable. Hence, the system output will tend to infinity, and the system will not reach a steady state. To be more precise, the general form of the solution to the differential equation describing the system output is $y(t) = C_0 e^t + \frac{1}{\sqrt{5}}\sin(2t - \pi + \arctan 2)$, and any initial state $y(0) \neq \frac{1}{\sqrt{5}}\sin(-\pi + \arctan 2)$ will lead to a solution that tends to infinity. This will almost always be the case in practice.

c) $0.11\sin(2t - 2.4)$
(Gain: $\left|\frac{1}{(\mathrm{i}2+1)(\mathrm{i}4+1)}\right| = \frac{1}{\sqrt{5}\sqrt{17}} \approx 0.11$, phase: $-\arg(\mathrm{i}2+1) - \arg(4\mathrm{i}+1) \approx -2.4$ rad $= -139°$.)

d) $0.45\sin(2(t-0.5) - 1.1) = 0.45\sin(2t - 2.1)$.
Similar to problem a), with an extra *time delay* of 0.5 s.

Go back

4.8 a) To determine the phase difference, ϕ, given a diagram with two sinusoids, $\sin(\omega t)$ and $K\sin(\omega t+\phi)$, one possibility is to consider the time points when the two curves pass 0. Determine t_1 and t_2 such that

$$\sin(\omega t_1) = 0$$
$$K\sin(\omega t_2 + \phi) = 0$$

This gives that $\omega t_1 = \omega t_2 + \phi$, that is,

$$\phi = -\omega t_\Delta$$

where $t_\Delta = t_2 - t_1$. For example, consider the second graph where $t_\Delta \approx 0.18$ s and $\omega = 5$ rad/s). Hence, $\phi = -0.9$ rad. This results in the table below, where the answer to part b is also included.

ω	$\lvert G(i\omega)\rvert$			$\arg G(i\omega)$		
1	1	=	0 dB_{20}	−0.2 rad	=	−11°
5	0.8	=	−1.9 dB_{20}	−0.9 rad	=	−52°
10	0.5	=	−6 dB_{20}	−1.6 rad	=	−92°
20	0.2	=	−14 dB_{20}	−2.2 rad	=	−126°

b) Just evaluate the decibel formula to obtain the values in the table above.

c) A Bode plot of the system is given in Figure 4.8a

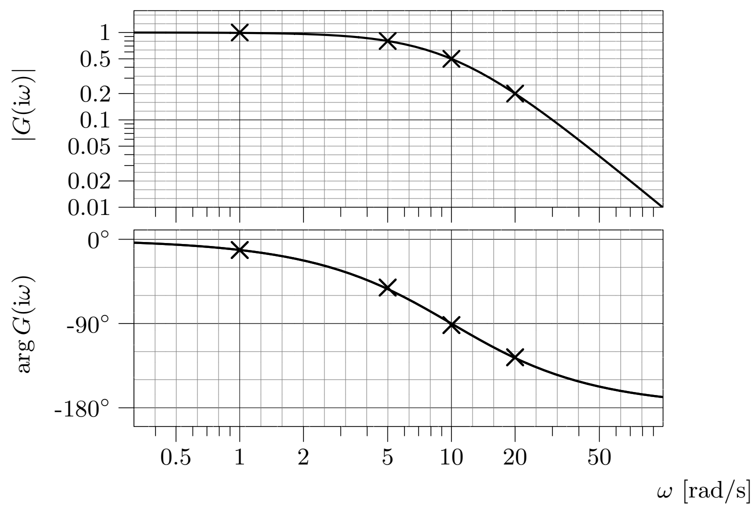


Figure 4.8a

Go back

4.9 **Answer:** G_1–B, G_2–D, G_3–A, G_4–C, G_5–E.

- The Bode plot B has static gain 1 and no resonance peak, and hence G_1–B. It can also be seen that the Bode plot B decays by one decade (20 dB_{20}) when the frequency increases by a factor of ten ("the slope is −1") and that G_1 has one pole.
- The Bode plots A and C have both infinite gain for when the frequency tends to zero, that is, they correspond to systems containing an integrator ⇒ systems G_3 and G_4. The Bode plot C decays more rapidly for high frequencies ⇒ the relative degree (number of poles − number of zeros) is higher. Hence G_3–A, G_4–C.

- The Bode plots D and E have peaks ⇒ systems G_2 and G_5. (For G_2 the peak is caused by the zero where the curve "turns up" at $\omega = 1$.) The Bode plot E has larger slope than D for high frequencies, that is, E corresponds to a system with higher relative degree. G_2 has one pole more than zeros, G_5 has 2 poles, and hence G_2–D, G_5–E.

Go back

4.10
- In *step response* i and iv the *step responses* tend to one, that is, they correspond to Bode gain A and C which have static gain 1. *Step response* iv has the largest *overshoot*, that is, it corresponds to Bode gain A with the largest resonance peak, and consequently *step response* i corresponds to Bode gain C. This gives the Bode gain–step response pairs i–C and iv–A.
- *Step response* ii has no *overshoot*, which implies that it corresponds to Bode gain D, which has no resonance peak. This gives the combination D–ii.
- The remaining combination is B–iii. *Step response* iii has an *overshoot* which can be related to the peak in the Bode gain plot. It can also be seen that this pair belongs to the fastest system, and we note that the static gain is lower than 1 and consequently the step converges to a value under 1.

Go back

4.11
a) The closed loop system a has a peak resonance, corresponding to an oscillatory system as in *step response* 1. An oscillatory system also means a system with small *phase margin* and complex poles.
Answer: B – a – 1 – II and A – b – 2 – I.

b) *Step response* 1 is faster than *step response* 2, that is, has a higher bandwidth and higher gain crossover frequency. Thus, it matches the closed loop system b and the open system B. A faster *step response* has the dominant pole further into the left half plane.
Answer: B – b – 1 – I and A – a – 2 – II.

Go back

4.12
a) Putting the denominator equal to zero gives the pole $s = -1$, and putting the numerator equal to zero gives the zero $s = -\alpha$.

b) The *static gain* of the system is given by $G(0)$, and in this case this gives

$$G(0) = 1$$

for all values of α.

c) The absolute value of the frequency function is given by

$$| G(i\omega) |= \frac{\sqrt{(\omega/\alpha)^2 + 1}}{\sqrt{\omega^2 + 1}} = \frac{\sqrt{(1/\alpha)^2 + 1/\omega^2}}{\sqrt{1 + 1/\omega^2}}$$

which tends to $1/\alpha$ as $\omega \to \infty$.

d) In both cases the curves are approximately horizontal at one for small values of ω and approximately horizontal at $1/\alpha$ for large values of ω. For $\alpha = 0.5$ the curve will "bend upwards" around $\omega = 0.5$ and then "bend back" around $\omega = 1$. For $\alpha = 2$ the curve will "bend down"

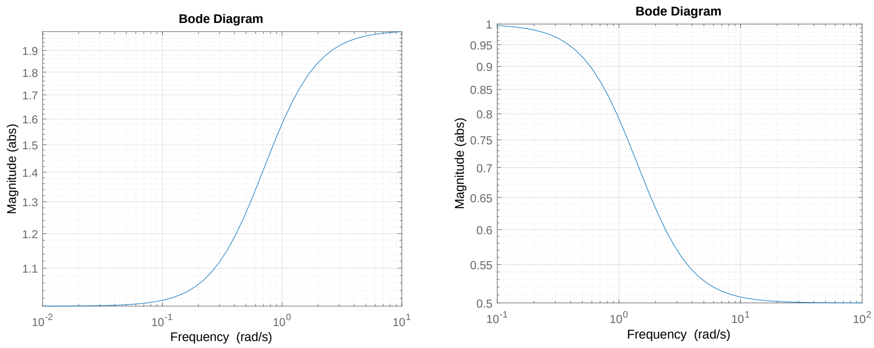


Figure 4.12a. $\mid G(i\omega) \mid$. *Left curve for* $\alpha = 0.5$. *Right curve* $\alpha = 2$.

around $\omega = 1$ and then "bend back" around $\omega = 2$. The curve hence bends down first when the zero is to the left of the pole on the negative real axis, while it bends up first when the zero is to the right of the pole. Figure 4.12a show the two cases. Both axes have logarithmic scales.

Go back

5 Compensation

5.1 Let G denote the heat exchanger's *transfer function*.

a) Draw the Bode plot using the given table. From the diagram in Figure 5.1a it follows that

$$\omega_c = 0.079 \text{ rad/s} \quad \varphi_m = 88^\circ \quad A_m = 5.0$$

b) A proportional controller does not change the phase curve. According to Figure 5.1a, the phase curve crosses -130° at the frequency 0.15 rad/s. A gain crossover at this frequency will yield exactly the required *phase margin*, and any higher *gain crossover frequency* will yield one that is too small.*

c) Twice as large *gain crossover frequency* is desired:

$$\omega_{c,d} = 0.30 \text{ rad/s} \quad \varphi_{m,d} = 50^\circ$$

At the frequency 0.30 rad/s the *phase margin* is -5°. Hence, a phase lead of 55° is needed. To this end, use the lead-compensator F_{lead}, where $F_{\text{lead}}(s) = K(\tau_D s + 1)/(\beta\tau_D s + 1)$. Set $\beta = 0.1$ (according to the diagram in Glad&Ljung) in order to achieve the required phase lead. To obtain the maximum phase lead at the desired gain crossover frequency, let

$$\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 10.54$$

Finally, K is chosen so that $\omega_{c,d}$ is obtained. From the data, we have $|G(\mathrm{i}0.3)| = 0.18$

$$1 = |F_{\text{lead}}(\mathrm{i}\omega_{c,d})G(\mathrm{i}\omega_{c,d})| = \frac{K}{\sqrt{\beta}}|G(\mathrm{i}\omega_{c,d})| \quad \Leftrightarrow K = \frac{\sqrt{\beta}}{0.18} = 1.76$$

*The controller gain that yields the desired *gain crossover frequency* can be computed as

$$K = \frac{1}{|G(0.15\mathrm{i})|} = \frac{1}{0.525} = 1.9$$

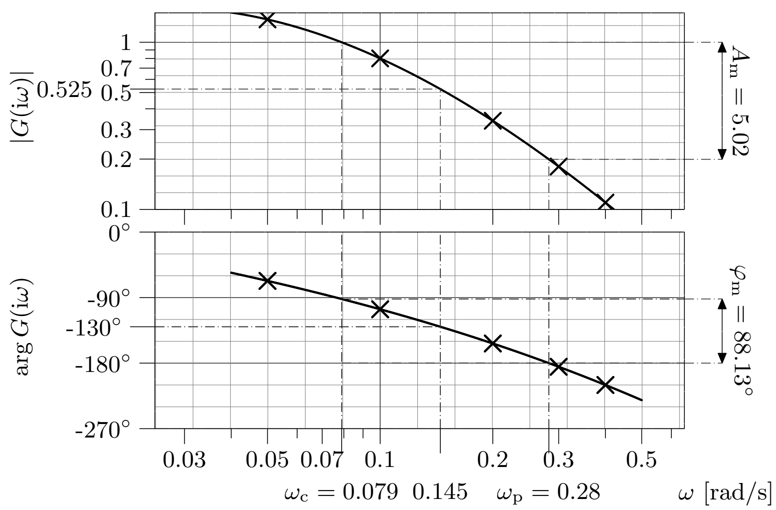


Figure 5.1a

Answer:

$$F(s) = 1.76\frac{(10.54s+1)}{(0.1\cdot 10.54s+1)}$$

5.2 *Notation.* The notation "A – B – C" is used to say that the system with open loop Bode plot in row A has its closed loop Bode plot in row B, and its *step response* in row C.

A good start is often to look at the *static gain* and the final value of the *step responses*. The *static gain* of the open loop system and the closed loop system are related as $|G_c(0)| = \frac{|G_o(0)|}{|1+G_o(0)|}$. Systems with the same *static gain* can then be separated by looking at stability margins, resonance peek, *overshoot*, bandwidth, and speed. Three of the combinations are easy to identify:

B – C – E: Infinite open loop *static gain* matches closed loop *static gain* equal 1, which in turn matches a *step response* that settles at amplitude 1.

C – A – B: Zero static *loop gain* matches zero closed-loop gain, which in turn matches a *step response* that settles at amplitude 0. It is also possible to relate the Bode plots by their high frequency gains.

A – E – C: Finite but non-zero open loop *static gain* matches non-zero closed loop *static gain* less than 1. No resonance frequency in neither the open-loop or closed-loop can be seen, and the *step response* is monotonic without any *overshoot*

The remaining open loop Bode plots are D and E. These should be matched with the closed-loop gain curves B and D, and *step responses* A and D. All Bode plots show signs of resonance frequencies in the open-loop, surving to the closed-loop. Both open loop Bode plots show a *static gain* near 1, which will make it hard (albeit possible) to use that feature for identification. We note though that D has a significantly lower bandwidth than E. Since the closed-loop is given by $G_c = \frac{G_o}{1+G_o}$ we can expect the closed-loop bandwidth with D to be much lower than with E (consider for instance a high frequency where open-loop E is around 1 in absolute value, at that point D will be very small, consequently the closed-loop terms will be around 1 and 0 in order of magnitudes). A high bandwidth in closed loop leads to a faster system. The last two combinations are thus D–D–D, E–B–A.

Go back

5.3

$$G(s) = \frac{10}{s(1+\frac{s}{20})(1+\frac{s}{40})(1+\frac{s}{100})}$$

gives

$$|G(i\omega)| = \frac{10}{\omega\sqrt{1+(\frac{\omega}{20})^2}\sqrt{1+(\frac{\omega}{40})^2}\sqrt{1+(\frac{\omega}{100})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \to \frac{10}{\omega},\ \omega \to 0$$

and

$$\arg G(i\omega) = -90^\circ - \arctan\frac{\omega}{20} - \arctan\frac{\omega}{40} - \arctan\frac{\omega}{100}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $\left|\frac{10}{10}\right| = 1$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		20		40		100	
Slope	−1		−2		−3		−4

The phase curve is drawn based on a couple of samples:

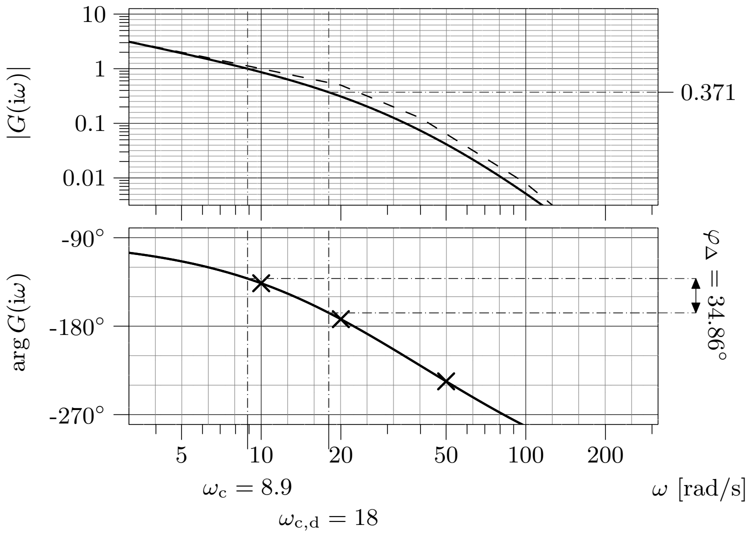


Figure 5.3a

Frequency [rad/s]	10	20	50
Phase	$-136°$	$-173°$	$-236°$

In addition, one can also use

$$\arg G(i\omega) \to -90°,\ \omega \to 0$$
$$\arg G(i\omega) \to -360°,\ \omega \to \infty$$

The Bode plot in Figure 5.3a gives that $\omega_c = 8.9$ rad/s, $\varphi_m = 48°$ and $A_m = 3.9$. However, it is only the *gain crossover frequency* which directly interests us here; an increase of the speed with a factor of two and a preserved damping imply $\omega_{c,d} = 18$ rad/s and $\varphi_{m,d} = \varphi_m$. From the figure, we have $\varphi_\Delta = \arg G(i\omega_c) - \arg G(i\omega_{c,d}) = 35°$. We suspect a lag compensator will be introduced later, as we have low frequency requirements. A lag compensator, if designed according to the prescribed recipe, will decrease the phase by at most 6° in the designed *gain crossover frequency*. Therefor, we require an additional 6° phase advancement and end up in 41° To this end, use a lead compensator (with standard notation of the parameters) with $\beta = 0.21$ and $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 0.12$. K is adjusted to get the desired gain crossover frequency:

$$|G(i\omega_{c,d})F_{\text{lead}}(i\omega_{c,d})| = |G(i\omega_{c,d})| \cdot \frac{K}{\sqrt{\beta}} = 1 \quad \Rightarrow \quad K = \frac{\sqrt{\beta}}{0.37} = 1.2$$

The *transfer function* from the reference input to the control error is given by

$$E(s) = \frac{1}{1 + F(s)G(s)}\theta_{\text{ref}}(s)$$

When $\theta_{\text{ref}}(t)$ is a step, the *final value theorem* gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

even without a lag compensator thanks to the integration in $G(s)$. Here, the *final value theorem* may be used since the system by construction is stable (the *phase margin* is positive).

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then $|F_{\text{lag}}(0)| = 1/\gamma$, and if $\theta_{\text{ref}}(t) = 10 \cdot t$, that is, if

$$\theta_{\text{ref}}(s) = \frac{10}{s^2}$$

one obtains

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+F(s)G(s)}\frac{10}{s^2} = \frac{10\gamma}{k_{\mathrm{m}} \cdot K} < 0.01$$

which gives $\gamma < 0.01K = 0.012$. Take $\gamma = 0.012$ to avoid excessively high low frequency loop gain. According to the rule of thumb, let $\tau_{\mathrm{I}} = 10/\omega_{\mathrm{c,d}} = 0.56$.

Answer:

$$F(s) = 1.2\frac{0.12s+1}{0.21 \cdot 0.12s+1} \cdot \frac{0.56s+1}{0.56s+0.012}$$

Go back

5.4

$$G(s) = \frac{1}{s}G_1(s)$$

gives

$$|G(\mathrm{i}\omega)| = \frac{|G_1(\mathrm{i}\omega)|}{\omega}$$
$$\arg G(\mathrm{i}\omega) = G_1(\mathrm{i}\omega) - 90^\circ$$

A P controller gives a phase margin of 40° when

$$\arg G(\mathrm{i}\omega) = -140^\circ \quad \Rightarrow \quad \arg G_1(\mathrm{i}\omega) = -50^\circ$$

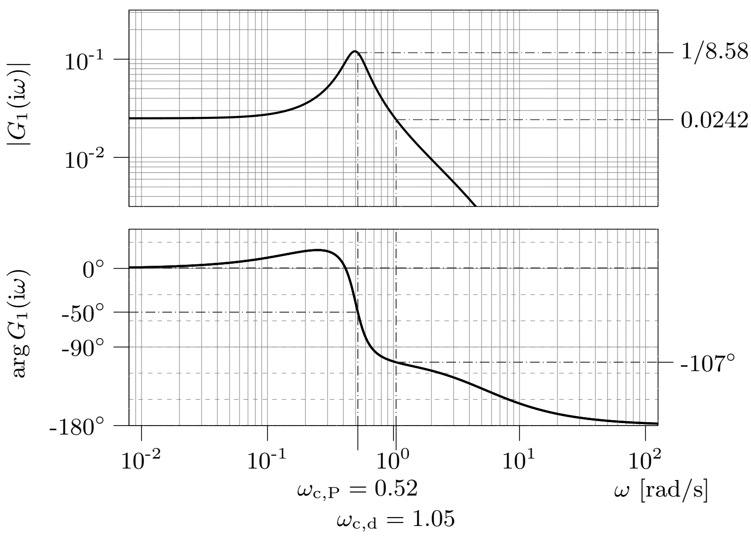


Figure 5.4a

From Figure 5.4a it is seen (although not easily) that this occurs at $\omega_{\mathrm{c,P}} = 0.52$ rad/s, which is also the highest possible gain crossover frequency possible to obtain with P control. The desired increase in speed by a factor of two is thus achieved by a new gain crossover frequency $\omega_{\mathrm{c,d}} = 1.05$ rad/s. Figure 5.4a gives

$$\arg G_1(\mathrm{i}\omega_{\mathrm{c,d}}) = -107^\circ \quad \Rightarrow \quad \arg G(\mathrm{i}\omega_{\mathrm{c,d}}) = -197^\circ$$

A desired phase margin of 40° requires that the phase be advanced by $57^\circ + 6^\circ = 63^\circ$. To this end, employ a two equal lead compensators (using standard notation of parameters), each advancing the phase by 32°; take $\beta = 0.31$ and $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 1.72$. The controller gain is adjusted by the factor K to get the desired gain crossover frequency:

$$|F_{\text{lead}}(i\omega_{c,d})|^2 \cdot |G(i\omega_{c,d})| = 1 \quad \Rightarrow \quad K^2 \frac{1}{\sqrt{0.31}^2} \frac{0.024}{1.05} = 1 \quad \Rightarrow \quad K = \sqrt{13.3}$$

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then $|F_{\text{lag}}(0)| = 1/\gamma$, and $|F(0)| = K^2/\gamma$. To choose γ, consider the Laplace transform of the control error,

$$E(s) = \frac{1}{1 + F(s)G(s)} R(s)$$

If $r(t) = A \cdot t$ (a ramp), that is, if

$$R(s) = \frac{A}{s^2}$$

one obtains

$$\begin{aligned}\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1 + F(s)G_1(s)/s}\frac{A}{s^2} = \lim_{s\to 0}\frac{A}{s + F(s)G_1(s)} \\ = \frac{A}{|F(0)| \cdot |G_1(0)|}\end{aligned}$$

This shows that the ramp error is inversely proportional to the *static gain* of the controller. According to Figure 5.4a, the highest possible controller gain when using a P controller and a *phase margin* of 40° is required, is $8.6\,\omega_{c,P} = 4.5$ (remember that $\frac{1}{s}$ contributes with $\frac{1}{\omega_{c,P}}$ to the *loop gain* at $\omega_{c,P}$). Hence, to reduce the ramp error to 1% of that of the P controller, the *static gain* of the new controller has to be at least 450. Therefore, take $\gamma = K^2/450 = 0.0296$, and, according to the rule of thumb, let $\tau_I = 10/\omega_{c,d} = 9.52$.

Answer:

$$F(s) = 13.3 \cdot \left(\frac{1.72s + 1}{0.31 \cdot 1.72s + 1}\right)^2 \frac{9.52s + 1}{9.52s + 0.0296}$$

Go back

5.5 a) *Gain crossover frequency* ω_c is defined as the frequency where the amplitude gain $|G(i\omega)|$ is 1, which appears to be around 0.8 rad/s.

Phase margin is the margin down to −180° at ω_c, i.e., $\arg G(i\omega_c) - (-180°)$. Since the phase is around −130° the phase margin is 50°. Consequently, the closed-loop $\frac{G(s)}{1+G(s)}$ is stable.

The *phase crossover frequency* ω_p is the frequency where the phase is −180° which thus is 3 rad/s.

The *gain margin* is $1/|G(i\omega_p)|$. Since $|G(i\omega_p)| = 0.1$ we conclude the gain margin is 10.

b) The *gain margin* is $1/|G(i\omega_p)| = 10$, which is also the largest possible proportional gain that preserves closed loop asymptotic stability. This follows as we now have lifted the whole gain curve by a factor 10 and have $|10G(i\omega_p)| = 1$. This means we will have a situation where the new loop-gain is 1 and the phase is −180° at ω_p which now also will be the gain crossover frequency, and thus on the border of instability as the phase margin now is 0°.

c) The Laplace transform of the control error is related to the reference as follows:

$$E(s) = \frac{1}{1 + KG(s)} R(s)$$

With

$$r(t) = 10t \quad \Rightarrow \quad R(s) = \frac{10}{s^2}$$

and using the *final value theorem* (from b we have that the system is stable), this yields

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{10}{2\lim_{s\to 0} sG(s)}$$

For small ω we have

$$G(s) \approx \frac{1}{s} \quad \Rightarrow \quad sG(s) \to 1,\ s \to 0 \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = 5$$

d) Raising the gain curve in the Bode plot by $K = 2$ results in

$$\omega_c = 1.24 \text{ rad/s} \quad \varphi_m = 32^\circ$$

The closed loop system becomes unstable when the *phase margin* is eaten up by the phase lag of the delay,

$$\arg e^{-i\omega T} = -\omega T$$

so in order to get an asymptotically stable closed loop system it is thus required that

$$\omega_c T < 32^\circ \quad \Rightarrow \quad T < \frac{32^\circ}{1.24 \text{ rad/s}} = \frac{0.55 \text{ rad}}{1.24 \text{ rad/s}} = 0.44 \text{ s}$$

e) The Nyquist curve is drawn based on the following observations: First, as $\omega \to 0$, $|G(i\omega)|$ increases and $\arg G(i\omega) \to -90^\circ$. Then, as $\omega \to \infty$, $|G(i\omega)| \to 0$ and $\arg G(i\omega)$ decreases. We also have, $\omega_c = 0.78$ rad/s with $\arg G(i\omega_c) = -133^\circ$, and finally $\omega_p = 3.2$ rad/s with $|G(i\omega_p)| = 0.091$. The resulting Nyquist curve is shown in Figure 5.5a.

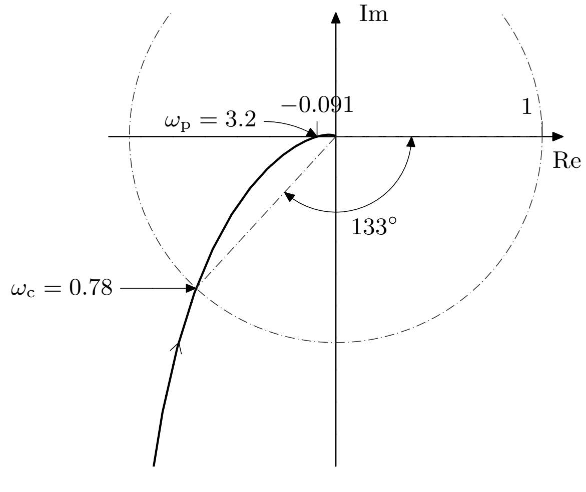


Figure 5.5a

Go back

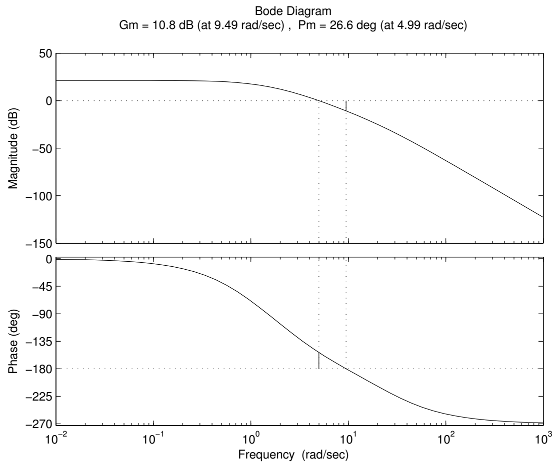
5.6 a) For this amplitude curve we cannot say anything about the stability since the system can have *arbitrarily* negative phase in any frequency.

b) It is stable, since the gain is less than 1 for all frequencies; there is no risk that the Nyquist curve could encircle −1 under these circumstances.

Go back

5.7 a) Enter the system and the regulator. Draw the Bode plot. This gives $\omega_c = 5$ rad/s, $\omega_p = 9.5$ rad/s, $A_m = 3.5$ and $\varphi_m = 27°$.

```
>> s = tf( 's' );
>> G = 725 / ...
      ( ( s + 1 ) * ( s + 2.5 ) * ( s + 25 ) );
>> F = 1;
>> margin( F * G )
```



b) From a) we know that at $\omega_{c,d} = 5$ rad/s the *phase margin* is 27°. In order to have $\varphi_m \geq 60°$ we need to increase the phase by approximately 40°, including 6° extra to compensate for a future lag compensator. This is obtained using a lead compensator (using standard notation of parameters) with $\beta = 0.21$. The phase compensation is located at the correct frequency by taking $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 0.43$.

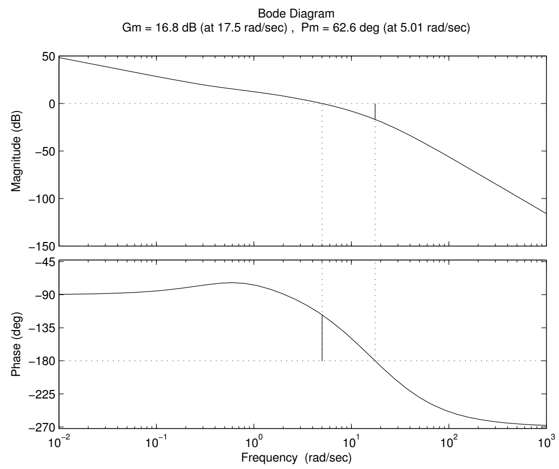
The controller gain is adjusted by the factor K to get the desired gain crossover frequency:

$$K \cdot \frac{1}{\sqrt{\beta}} \cdot |G(i5)| = K \cdot \frac{1}{\sqrt{0.21}} \cdot 1 = 1 \quad \Rightarrow$$
$$K = 0.46$$

The requirement $e_0 = 0$, that is, no *steady state error* for a *unit step* reference signal, is achieved by incorporating a lag compensator (using standard notation of parameters) with $\gamma = 0$, and, using the rule of thumb for the choice of τ_I, we take $\tau_I = 10/5 = 2$.

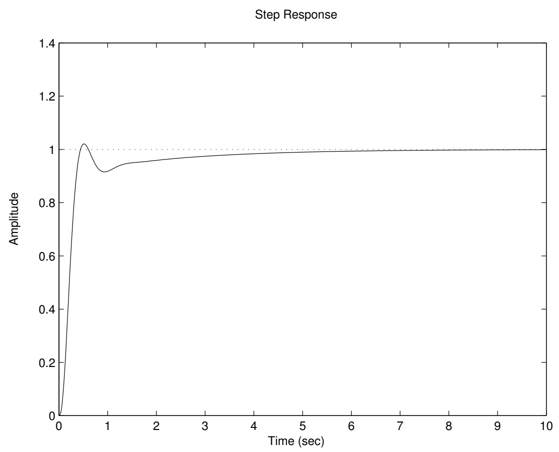
Generate a lead-lag regulator and make a Bode plot of the open loop system. Both the *gain crossover frequency* and the *phase margin* requirements are satisfied.

```
>> wc = 5;
>> b = 0.21;
>> tD = 1 / ( wc * sqrt( b ) );
>> K = sqrt( b ) / 1;
>> Flead = K*( tD * s + 1 ) / ( b * tD * s + 1 );
>> g = 0;
>> tI = 10 / wc;
>> Flag = ( tI * s + 1 ) / ( tI * s + g );
>> F = Flead * Flag;
>> margin( F * G )
```



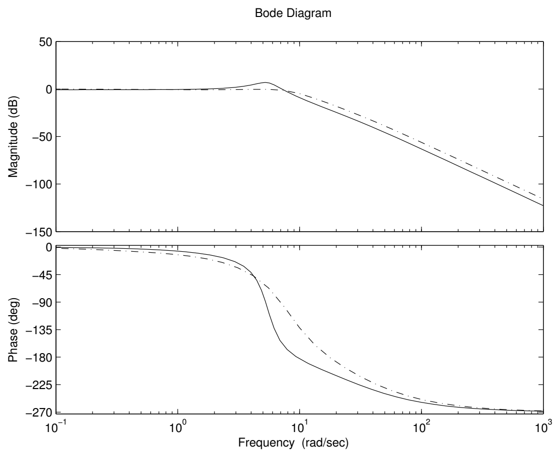
Plot the *step response* of the closed loop system.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 10 )
```



c) Compute the *transfer function* of the closed loop system for $F(s) = 1$. Draw its Bode plot side by side with the Bode plot for the compensated system. (The curves of the compensated system are dash-dotted.)

```
>> Gc1 = feedback( G, 1 );
>> bode( Gc1, '-', Gc, '-.' )
```



Comparing the two Bode plots we see that the main difference is that the height of the resonance peak has been reduced, that is, the damping of the closed loop system has been increased due to the increased *phase margin*. We also see that the bandwidth is approximately the same, since we have not changed the *gain crossover frequency*.

d) Calculate the *transfer function* from the reference signal to the error:

$$E(s) = R(s) - F(s)G(s)E(s) \quad \Rightarrow \quad E(s) = \frac{1}{1 + F(s)G(s)} R(s)$$

Let

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

Enter the *transfer function* S.

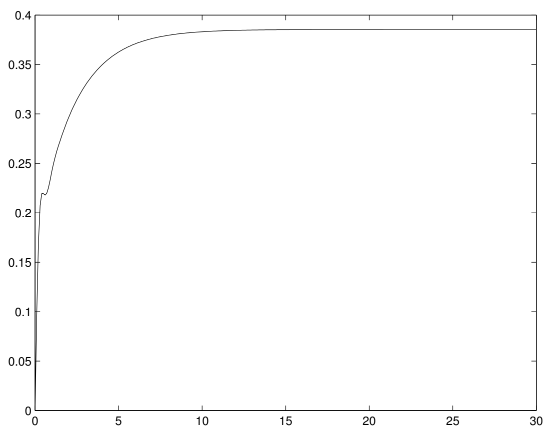
```
>> S = 1 / ( 1 + F * G );
```

Create a time vector between 0 and 30 with step 0.1, and a reference signal vector $r(t) = t$.

```
>> t = ( 0 : 0.1 : 30 ).';
>> r = t;
```

Plot the result. Even though the *steady state error* for a step reference signal is zero (due to $\gamma = 0$), the *steady state error* for a ramp reference signal is non-zero.

```
>> y = lsim( S, r, t );
>> plot( t, y )
```



Go back

5.8 a) The phase curve crosses $-120°$ at $\omega = 0.27$ rad/s and there the gain is $|\ G_0(i\omega)\ | \approx 0.35$. The maximum gain crossover frequency is hence $\omega_c = 0.27$ rad/s and it is achieved for $K = 1/0.35 = 2.86$.

b) The reference signal $r(t) = 0.5t$ implies that $R(s) = 0.5/s^2$. The steady state error can be computed using the final value theorem, which implies

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+\frac{K\ \cdot\ 0.1}{s(s+1)^2}}\frac{0.5}{s^2} = \frac{5}{K} \approx 1.75$$

c) In order to reduce the steady state error it will be necessary to introduce a lag compensation, but in order to maintain phase margin 60° after the lag compensation has been included it will be necessary to introduce a lead compensation. Introduce the lead compensation

$$F_{lead}(s) = K\frac{\tau_D s+1}{\beta\tau_D s+1}.$$

In order to guarantee phase margin 60° the phase needs to be increased by $\approx 6° \Rightarrow \beta = 0.7$. With $\omega_{c,d} = 0.27$ rad/s from problem a) this implies that $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 4.44$. The gain K is given by the relationship

$$|F_{lead}(i\omega_{c,d})G(i\omega_{c,d})| = K\frac{1}{\sqrt{\beta}}0.35 = 1$$

which gives $K = 2.39$

In order to reduce the steady state error we introduce

$$F_{lag}(s) = \frac{\tau_I s+1}{\tau_I s+\gamma}$$

The requirement on the steady state error, using the same reference signal as in b), gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+F_{lead}(s)F_{lag}(s)G(s)}\frac{0.5}{s^2} = \frac{\gamma}{K}5 \le 0.175$$

which implies that $\gamma \le 0.175\ \cdot\ K/5 \approx 0.1$. Using the rule of thumb from Glad&Ljung implies $\tau_I = 10/\omega_c = 37$. The entire feedback controller hence becomes

$$F(s) = F_{lead}(s)F_{lag}(s) = 2.39\frac{(4.4s+1)}{(0.7\ \cdot\ 4.4s+1)}\frac{(37s+1)}{(37s+0.1)}$$

Go back

5.9 In B and C the gain of $G_O(i\omega)$ tends to infinity when ω tends to zero, which means that they can be combined with I and II since these curves have steady state gain one, i.e. $G_C(0) = 1$. The curve in B has higher gain crossover frequency and lower phase margin, which implies that it corresponds to I, which has the highest resonance peak. This gives the combinations B - I and C - II respectively. Using the same arguments it can be seen that A has higher gain crossover frequency and lower phase margin, which means that it correponds to III. Hence D corresponds to IV.

Go back

5.10 a) The phase for low frequencies tends to $-90°$ which implies that the system contains an integrator, i.e. a factor s in the denominator, which means that $p = 1$. For high frequencies the phase tends to $-270°$ which means that the difference between the order of the denominator and the order of the numerator is three, i.e. $p + n - m = 3$.

b) Use a lead-lag compensator, i.e.

$$F = K \frac{\tau_D s + 1}{\tau_D \beta s + 1} \frac{\tau_I s + 1}{\tau_I s + \gamma}.$$

At the desired gain crossover frequency $\omega_{\mathrm{c,d}} = 3$ we have $\arg G(i3) \approx -178°$ and $|G(i3)| = 0.1$. In order to obtain phase margin 45°, also including a lag compensator, the phase has to be increased by $45 - 2 + 6 = 49°$. This is obtained by choosing

$$\beta = 0.13, \quad \tau_D = \frac{1}{\omega_{\mathrm{c,d}}\sqrt{\beta}} = 0.92.$$

Take also $K = 3.6$ such that

$$|F_{lead}(i\omega_{\mathrm{c,d}}| \cdot |G(i\omega_{\mathrm{c,d}})| = \frac{K}{\sqrt{\beta}} \cdot 0.1 = 1.$$

For low frequencies the transfer function can be approximated by

$$G(s) \approx \frac{A}{s}$$

where the Bode diagram, by looking at $\omega = 0.01$, gives that $A = 1$.

The steady state error when $r(t)$ is a unit step becomes

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1 + FG(s)} \frac{1}{s} = \lim_{s\to 0} \frac{1}{1 + \frac{K}{\gamma}\frac{1}{s}} = 0.$$

and the steady state error when $r(t)$ is a unit ramp becomes

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1 + FG} \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{\frac{K}{\gamma}\frac{1}{s}s} = \frac{\gamma}{K},$$

By choosing

$$\gamma = 0.01K = 0.036,$$

and

$$\tau_I = \frac{10}{\omega_{\mathrm{c,d}}} = 3.3.$$

according to the rule of thumb we get

$$F(s) = 3.6 \frac{(0.92s + 1)}{(0.92 \cdot 0.13s + 1)} \frac{(3.3s + 1)}{(3.3s + 0.036)}.$$

5.11 B and C have a smaller stationary error than A and D $\Rightarrow$ Higher K $\Rightarrow$ B and C $\leftrightarrow$ *i,ii*, A and D $\leftrightarrow$ *iii,iv*. Small $\beta \Rightarrow$ Large increase in phase margin $\Rightarrow$ Less oscillatory $\Rightarrow$ A and C $\leftrightarrow$ *i,iii*, B and D $\leftrightarrow$ *ii,iv*.

Answer: A – *iii*, B – *ii*, C – *i* and D – *iv*.

6 Sensitivity and Robustness

6.1 The *sensitivity function* is the *transfer function* from v to y. The block diagram gives

$$Y(s) = \frac{1}{1+\frac{K}{s(s+1)}}V(s) = \underbrace{\frac{s^2+s}{s^2+s+K}}_{S(s)}V(s)$$

$$|S(\mathrm{i}\omega)| = \frac{\omega\sqrt{\omega^2+1}}{\sqrt{(K-\omega^2)^2+\omega^2}}$$

For $\omega = 1$ we get

$$|S(1\mathrm{i})| = \frac{\sqrt{2}}{\sqrt{(K-1)^2+1}}$$

The amplitude of $y(t)$ is less than the amplitude of $v(t)$ if $|S(1\mathrm{i})| < 1$, that is,

$$\frac{\sqrt{2}}{\sqrt{(K-1)^2+1}} < 1 \quad \Leftrightarrow \quad 2 < (K-1)^2+1 \quad \overset{K>0}{\Leftrightarrow} \quad K > 2$$

Go back

6.2 Determine the upper limit of the relative model error

$$\Delta(s) = \frac{G^0(s)-G(s)}{G(s)} = s \quad \Rightarrow \quad |\Delta(\mathrm{i}\omega)| = \omega$$

The stability is then guaranteed if

$$|G_\mathrm{c}(\mathrm{i}\omega)| = \left|\frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1+F(\mathrm{i}\omega)G(\mathrm{i}\omega)}\right| < \frac{1}{\omega} \qquad \forall\omega$$

No *steady state error* for steps implies $G_\mathrm{c}(0) = 1$ and the bandwidth ω_B is thus defined by the smallest value that satisfies

$$|G_\mathrm{c}(\mathrm{i}\omega)| < \frac{1}{\sqrt{2}}, \qquad \omega > \omega_\mathrm{B}$$

The curve $1/\omega$ crosses $1/\sqrt{2}$ at $\omega = \sqrt{2}$. Thus, the bandwidth must be less than $\sqrt{2}$. However, the curve $|G_\mathrm{c}(\mathrm{i}\omega)|$ asymptotically approaches a line with slope $-20\ \mathrm{dB}_{20}$/decade, which implies that ω_B cannot be *arbitrarily* close to $\sqrt{2}$.

For example, if G_c is a first order system, then the breakpoint of the asymptote must be 1 rad/s if it shall coincide with $1/\omega$. The first order system with that asymptote is $\frac{1}{1+s/1}$, which has a bandwidth of 1 rad/s. If G_c would be a higher order system, the bandwidth could be made slightly higher, but the limited information about G_c excludes this possibility.

Answer: The maximum bandwidth is $\omega_\mathrm{B} = 1$.

Go back

6.3 The *disturbance* is amplified when the magnitude of the *sensitivity function* exceeds one, that is, when

$$\left|\frac{1}{1+G_\mathrm{o}(\mathrm{i}\omega)}\right| > 1$$

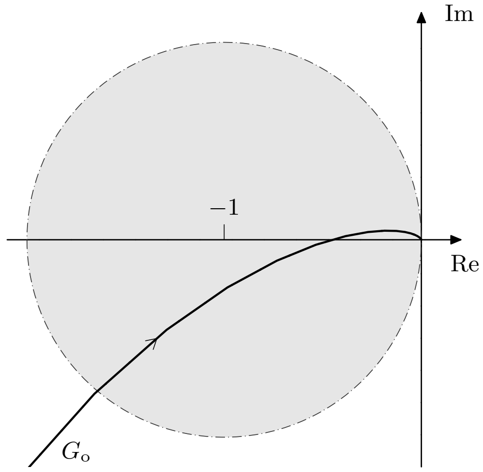


Figure 6.3a

that is

$$|1 + G_o(i\omega)| < 1$$

which corresponds to the part of $G_o(i\omega)$ that is within a circle with center at -1 and radius 1, see Figure 6.3a.

Go back

6.4 Let

$$g(\omega) = \frac{0.9}{\sqrt{1+\omega^2}}$$

denote the upper bound on the norm of the relative model error. Robustness condition:

$$|T(i\omega)| = \left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right| < \frac{1}{g(\omega)} \qquad \forall\omega$$

Now,

$$F(s)G(s) = \frac{s+10}{s}\frac{1}{s+10} = \frac{1}{s} \quad \Rightarrow$$

$$\left|\frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)}\right| = \left|\frac{1}{i\omega+1}\right| = \frac{1}{\sqrt{\omega^2+1}}$$

so the robustness condition becomes

$$\forall\omega: \ \frac{1}{\sqrt{\omega^2+1}} < \frac{\sqrt{\omega^2+1}}{0.9} \quad \Leftrightarrow$$

$$\forall\omega: \ 0.9 < \omega^2 + 1$$

which is satisfied. **Answer:** Yes.

Go back

6.5 a) Using notation similar to that in Glad&Ljung, we have

$$\Delta(s) = e^{-sT} - 1$$

that is, $\Delta(\mathrm{i}\omega) = \cos\omega T - 1 - \mathrm{i}\sin\omega T$. This implies

$$|\Delta(\mathrm{i}\omega)| = \sqrt{2 - 2\cos\omega T}$$

and in particular

$$|\Delta(\mathrm{i}\omega)| = \begin{cases} 0, & \text{when} \cos\omega T = 1 \\ 2, & \text{when} \cos\omega T = -1 \end{cases}$$

In Figure 6.5a, $|\Delta(\mathrm{i}\omega)|^{-1}$ is plotted as a function of ωT.

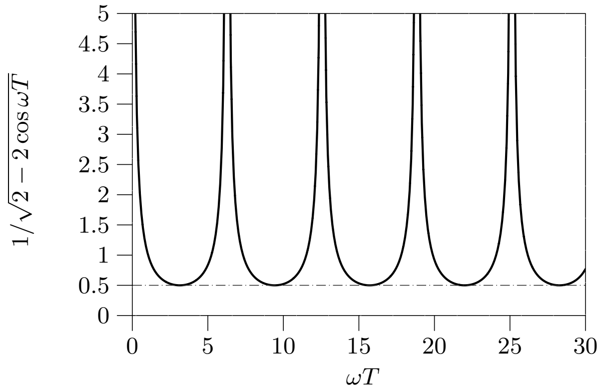


Figure 6.5a

b) The robustness criterion results in

$$\forall\omega : \left|\frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1 + F(\mathrm{i}\omega)G(\mathrm{i}\omega)}\right| < \frac{1}{|\Delta(\mathrm{i}\omega)|}$$

Figure 6.5a therefore provides the answer.

Answer:

$$\left|\frac{F(\mathrm{i}\omega)G(\mathrm{i}\omega)}{1 + F(\mathrm{i}\omega)G(\mathrm{i}\omega)}\right| < \frac{1}{2}$$

Go back

6.6 a) The characteristic equation can be determined for a generic nominal *loop gain*. Note that you are not required to derive the generic formula — just make sure that you are able to determine the correct polynomials $P(s)$ and $Q(s)$ below. Let

$$G_\mathrm{o}(s) = \frac{b(s)}{a(s)}$$

denote the nominal *loop gain*. The true closed loop system becomes

$$G_\mathrm{c}(s) = \frac{\frac{b(s)}{a(s)}\frac{\alpha}{s+\alpha}}{1 + \frac{b(s)}{a(s)}\frac{\alpha}{s+\alpha}} = \frac{b(s)\alpha}{a(s)(s+\alpha) + b(s)\alpha} = \frac{b(s)\alpha}{a(s)s + (a(s) + b(s))\alpha}$$

and has the same *root locus* with respect to α as the open loop system

$$\frac{a(s) + b(s)}{a(s)s} = \frac{G_\mathrm{o} + 1}{s}$$

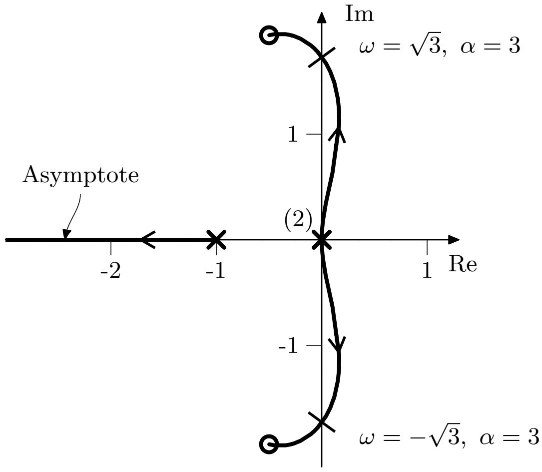


Figure 6.6a

has with respect to a proportional feedback. This can be used to draw the root locus using MATLAB. However, to draw the root locus by hand, we use that here $G_o(s) = KG(s)$, so

$$b(s) = 4 \qquad a(s) = s(s+1)$$

which lets us identify the polynomials P and Q in the characteristic equation $P(s)+\alpha Q(s) = 0$ as

$$P(s) = a(s)s = s^2(s+1) \qquad Q(s) = a(s) + b(s) = s^2 + s + 4$$

- Starting points ⇒ zeros of $P(s)$: 0 (double), and -1
  End points ⇒ zeros of $Q(s)$: $-\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{15}}{2}$
- Number of asymptotes: $3 - 2 = 1$.
  Direction of asymptote: $\frac{1}{1} \cdot \pi$, that is, the negative real axis.
- Part of the real axis that belongs to the root locus: $(-\infty, -1]$.
- Intersection with the imaginary axis: Set $s = \mathrm{i}\omega$ and solve the characteristic equation:

$$-\omega^2(\mathrm{i}\omega + 1) + \alpha(-\omega^2 + \mathrm{i}\omega + 4) = 0$$

  Isolate real and imaginary parts:

$$\begin{cases} -\omega^2(1+\alpha) + 4\alpha = 0 \\ -\omega^3 + \alpha\omega = 0 \end{cases}$$

  with solutions

$$(\,\alpha = 0,\, \omega = 0\,) \quad \text{or} \quad (\,\alpha = 3,\, \omega = \pm\sqrt{3}\,)$$

The root locus is shown in Figure 6.6a, from which the conclusion immediately follows.

Answer: Asymptotically stable for $\alpha > 3$.

b) Begin by identifying the relative model error:

$$G^0(s) = G(s)\frac{\alpha}{(s+\alpha)} = G(s)\Bigg(1 + \underbrace{\frac{\alpha}{(s+\alpha)} - 1}_{\Delta(s)}\Bigg)$$

Thus

$$\frac{1}{|\Delta(\mathrm{i}\omega)|} = \left|\frac{s+\alpha}{-s}\right| = \frac{\sqrt{\omega^2+\alpha^2}}{\omega} =: f(\omega)$$

The robustness criterion $\forall\omega : \ |G_\mathrm{c}(\mathrm{i}\omega)| < f(\omega)$ is fulfilled if the low frequency asymptote of $f(\omega)$ exceeds the resonance peak at $\omega = 2$, where $|G_\mathrm{c}(\mathrm{i}2)| = 2$. This gives the condition

$$\frac{\sqrt{4+\alpha^2}}{2} > 2 \quad \overset{\alpha\geq 0}{\Leftrightarrow} \quad \alpha > \sqrt{12}$$

Answer: $\alpha > \sqrt{12}$

c) The robustness criterion gives a sufficient but not necessary condition, that is, the system can be stable even if the criterion is not satisfied. In this case for $3 < \alpha < \sqrt{12}$. With a *root locus* we obtain an exact characterization of the stabilizing parameter values, that is, a necessary and sufficient condition.

Go back

6.7 Since the equation for G_c has the same "F" in the numerator and the denominator, it follows that the *complementary sensitivity function* T and G_c are the same. It can be shown that both $F(\mathrm{i}\omega)G(\mathrm{i}\omega)$ and $F(\mathrm{i}\omega)G^0(\mathrm{i}\omega)$ tend to 0 as $\omega \to \infty$. The robustness criterion guarantees stability if $|T(\mathrm{i}\omega)| < 1/(\gamma\omega)$ since

$$|\Delta(\mathrm{i}\omega)| < \gamma\omega \quad \Rightarrow \quad \frac{1}{\gamma\omega} < \frac{1}{|\Delta(\mathrm{i}\omega)|}$$

The *transfer function* T has a resonance peak at $\omega = 1$ (seen in Figure 6.7a, since $T = G_\mathrm{c}$) with $|T(\mathrm{i}1)| = 35$, which leads to the condition

$$35 < \frac{1}{\gamma \cdot 1} \quad \Leftrightarrow \quad \gamma < \frac{1}{35}$$

Trivially, γ must also be positive.

Answer: $0 \leq \gamma < \frac{1}{35}$

Go back

6.8 The closed loop system becomes

$$Y(s) = V(s) + G_\mathrm{o}(s)(R(s) - N(s) - Y(s)) \qquad \Rightarrow$$

$$Y(s) = \frac{G_\mathrm{o}(s)}{1+G_\mathrm{o}(s)}(R(s) - N(s)) + \frac{1}{1+G_\mathrm{o}}V(s)$$

where we can identify

$$T(s) = \frac{G_\mathrm{o}(s)}{1+G_\mathrm{o}(s)} \qquad S(s) = \frac{1}{1+G_\mathrm{o}(s)}$$

Notice that $S(s) + T(s) = 1$. In the problem formulation we have $Y(s) = S(s)V(s)$ since the other inputs are zero. Hence, for $v(t) = \sin t$, we have

$$\mathcal{L}^{-1}\{SV\}(t) = \frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4})$$

and thus for $n(t) = \sin t$,

$$Y(s) = -T(s)N(s) = -(1 - S(s))N(s) = S(s)N(s) - N(s) \quad \Rightarrow$$

$$y(t) = \frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4}) - \sin(t)$$

Go back

6.9 a) Putting

$$G^0(s) = G(s)\frac{1}{(s+1)} = G(s)(1+\Delta(s))$$

gives

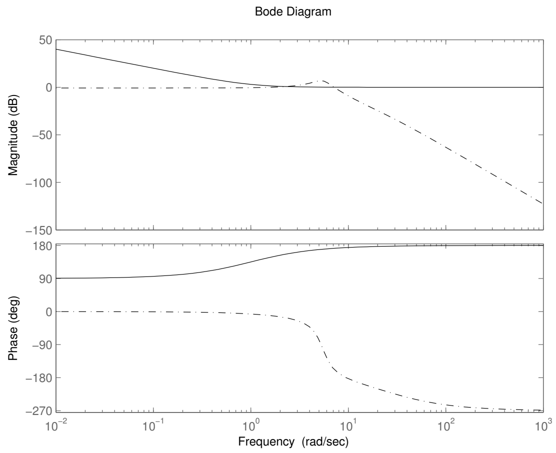
$$\Delta(s) = -\frac{s}{s+1} \quad \Leftrightarrow \quad \frac{1}{\Delta(s)} = -\frac{s+1}{s}$$

b) Enter the system and the regulator

```
>> s = tf( 's' );
>> G = 725 / ...
       ( ( s + 1 ) * ( s + 2.5 ) * ( s + 25 ) );
 >> F = 0.46*(0.43 s + 1 )*(2*s+1)/(2*s*(0.09*s+1));
```

Enter the inverse relative model error and the *complementary sensitivity function* obtained when $G(s)$ is controlled by $F(s) = 1$. Plot the amplitude curve of the inverse relative model error in the same diagram as the amplitude curve of the *complementary sensitivity function*.

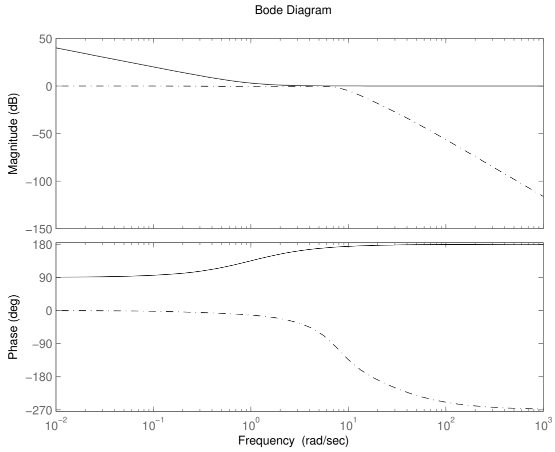
```
>> IDG = - ( s + 1 ) / s;
>> T = feedback( G*1 , 1 );
>> bode( IDG, 'k-', ...
         T, 'k-.' );
```



Since the absolute value of the *complementary sensitivity function* goes above the inverse relative model error over a frequency interval, we cannot guarantee that the closed loop system obtained when $G^0(s)$ is controlled by $F(s) = 1$ is asymptotically stable.

Enter the *complementary sensitivity function* obtained when $G(s)$ is controlled by the more advanced controller. Plot the amplitude curve of the inverse relative model error in the same diagram as the the amplitude curve of the *complementary sensitivity function*.

```
>> T = feedback( G*F, 1 );
>> bode( IDG, 'k-', ...
         T, 'k-.' );
```



In this case $|T(i\omega)|$ stays below the inverse relative model error, and hence we can guarantee that the closed loop system obtained when the advanced controller is applied to $G^0(s)$ will be asymptotically stable.

Go back

7 Special Controller Structures

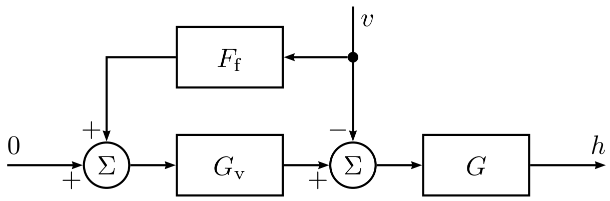


Figure 7.1a

7.1 Consider the block diagram in Figure 7.1a. The change in tank volume per time unit is given by

$$A\frac{\mathrm{d}}{\mathrm{d}t}h(t) = x(t) - v(t)$$

or, equivalently,

$$A \cdot s \cdot H(s) = X(s) - V(s)$$

which gives

$$H(s) = \frac{1}{As}(X(s) - V(s))$$

Furthermore,

$$X(s) = G_\text{v}(s)U(s)$$

where

$$G_\text{v}(s) = \frac{1}{1 + s/2}$$

a) We let the input $u(t)$ be a function of $v(t)$ only, that is,

$$U(s) = F_\text{f}(s)V(s)$$

The level $h(t)$ as a function of $v(t)$ then becomes

$$H(s) = \frac{1}{As}(G_\text{v}(s)F_\text{f}(s) - 1)V(s)$$

If we choose

$$F_\text{f}(s) = \frac{1}{G_\text{v}(s)} = 1 + s/2$$

the level becomes independent of $v(t)$, but to get the controller Stu uses, we remove the derivative term:

$$F_\text{f}(s) = 1$$

The level as a function of $v(t)$ then becomes

$$H(s) = \frac{1}{As}(\frac{1}{1 + s/2} - 1)V(s) = -\frac{1}{2A}\frac{1}{1 + s/2}V(s)$$

With $V(s) = 0.1/s$ this yields

$$H(s) = -\frac{0.1}{2A}\frac{1}{s(1 + s/2)} = -\frac{0.1}{2A}\left(\frac{1}{s} - \frac{1}{2 + s}\right)$$

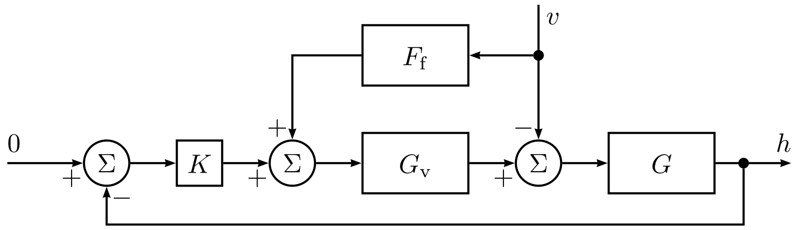


Figure 7.1b

that is

$$h(t) = -\frac{0.1}{A \cdot 2}(1 - e^{-2t})$$

which gives the steady state error $-0.05/A$.

b) We now choose the input $u(t)$ to be a function of both $h(t)$ and $v(t)$, that is, we add the term $-Kh(t)$ to the control law from a). (See Figure 7.1b.) Thus

$$u(t) = -Kh(t) + v(t)$$

or, equivalently,

$$U(s) = -KH(s) + V(s)$$

This gives

$$\begin{aligned} AsH(s) &= G_v(s)(-KH(s) + V(s)) - V(s) \\ (As + KG_v(s))H(s) &= (G_v(s) - 1)V(s) \\ \frac{H(s)}{V(s)} &= \frac{-s/2}{A/2 \cdot s^2 + As + K} = \frac{-s}{A(s^2 + 2s + 2K/A)} \end{aligned}$$

To select K, we may compare*

$$s^2 + 2s + 2K/A = 0$$

with the standard equation

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

which gives

$$\omega_0^2 = 2 \cdot K/A \qquad \zeta\omega_0 = 1$$

To obtain approximately 5% overshoot we choose $\zeta = 0.707$, and from

$$\sqrt{A/(2K)} = \zeta = 0.707$$

we get $K = A$. Hence,

$$H(s) = \frac{-s}{A(s^2 + 2s + 2)}V(s)$$

If $v(t)$ is a step of amplitude 0.1, the final level becomes

$$\lim_{t\to\infty} h(t) = \lim_{s\to 0} sH(s) = s\frac{-s}{A(s^2 + 2s + 2)}\frac{0.1}{s} = 0$$

that is, there will be no steady state error in the level for a step disturbance.

Go back

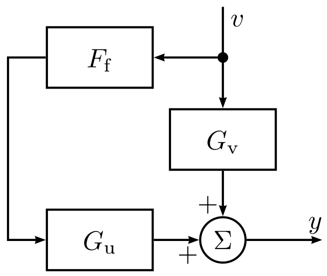


Figure 7.2a

7.2 a) A block diagram of the system is given in Figure 7.2a. The output is given by

$$Y = (G_v + G_u F_f)V$$

where

$$G_u(s) = \frac{2}{s+3} \qquad G_v(s) = \frac{3}{s+4}$$

Chose F_f such that $(G_v + G_u F_f)V = 0$:

$$F_f = -\frac{G_v}{G_u} = -\frac{3(s+3)}{2(s+4)}$$

Compute the controller.

```
>> s = tf( 's' );
>> Gu = 2 / ( s + 3 );
>> Gv = 3 / ( s + 4 );
>> F = - Gv / Gu;
```

b) If $v(t) = 2\sin\omega t$ then

$$u(t) = 2\,|F_f(i\omega)|\sin(\omega t + \arg F_f(i\omega))$$

The amplitude is then

$$A(\omega) = 2\,|F_f(i\omega)| = 2 \cdot \frac{3}{2}\sqrt{\frac{\omega^2+9}{\omega^2+16}} \leq 3$$

$A(\omega) \to 3, \ \omega \to \infty$

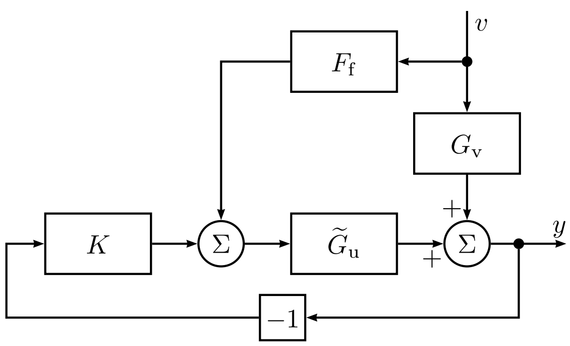


Figure 7.2b

*Note that any $K > 0$ results in a stable closed loop system, and that the steady state error computations below are independent of the particular value of K. Hence, selecting K is not necessary for the solution of this problem.

c) A block diagram of the system with both feedforward and *feedback* is shown in Figure 7.2b. The output is now given by

$$Y = G_v V + \widetilde{G}_u U = (G_v + \widetilde{G}_u F_f)V - \widetilde{G}_u K Y$$

where

$$\widetilde{G}_u(s) = \frac{b}{s+3}$$

The *transfer function* from V to Y is given by

$$\begin{aligned} Y(s) &= \frac{G_v + \widetilde{G}_u F_f}{1 + \widetilde{G}_u K} V(s) = \frac{\frac{3}{s+4} - \frac{3b}{2(s+4)}}{1 + K\frac{b}{s+3}} V(s) \\ &= \frac{3(1 - b/2)(s+3)}{(s+4)(s+3) + Kb(s+4)} V(s) \end{aligned}$$

This is stable for $K \geq 0$ and $b \geq 0$. The *final value theorem* can therefore be used (with $V(s) = \frac{1}{s}$):

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{3(1 - b/2)(s+3)}{(s+4)(s+3) + Kb(s+4)} \cdot \frac{1}{s} = \frac{9(1 - b/2)}{12 + 4Kb}$$

Go back

7.3 a) The output is given by

$$Y = (G_v + G_u F_f)V$$

where

$$G_u(s) = \frac{3}{s+1} \qquad G_v(s) = \frac{4}{(s+2)(s+5)}$$

Chose F_f such that $(G_v + G_u F_f)V = 0$:

$$F_f(s) = -\frac{G_v(s)}{G_u(s)} = -\frac{4(s+1)}{3(s+2)(s+5)}$$

Create the system and the feedforward controller.

```
>> s = tf( 's' );
>> Gv = 4 / ( s + 2 ) / ( s + 5 );
>> Gu = 3 / ( s + 1 );
>> F = - Gv / Gu;
```

b) The constant to replace $F_f(s)$ is given by

$$\widetilde{F}_f = F_f(0) = -\frac{4}{30}$$

The output is then given by

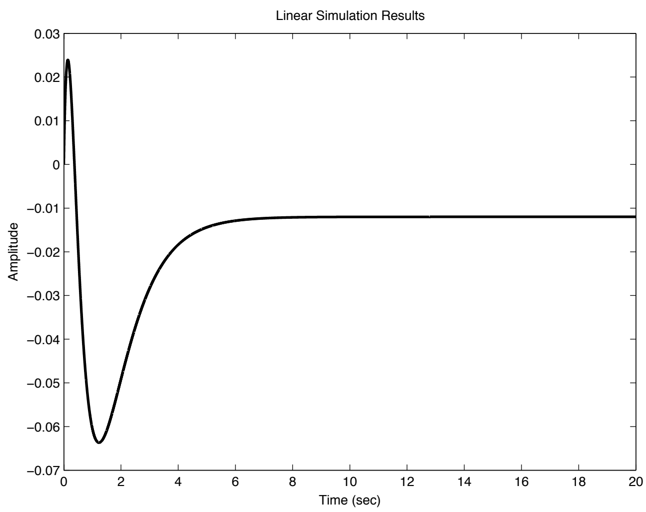
$$Y(s) = \left(-\frac{12}{30(s+1)} + \frac{4}{(s+2)(s+5)}\right)V(s) = \frac{40(s+1) - 4(s+2)(s+5)}{10(s+1)(s+2)(s+5)}V(s)$$
$$= \frac{-4s^2 + 12s}{10(s+1)(s+2)(s+5)}V(s)$$

Taking the Laplace transform of $v(t) = -1-0.1t$ we get $V(s) = -\frac{1}{s}-\frac{0.1}{s^2}$. The *final value theorem* then gives (verify that the system is stable)

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{-4s^2 + 12s}{10(s+1)(s+2)(s+5)}\left(-\frac{1}{s} - \frac{0.1}{s^2}\right)$$
$$= \frac{12}{100} \cdot (-0.1) = -0.012$$

Create the system with the controller and create the *disturbance* signal.

```
>> F = -4/30;
>> G = F * Gu + Gv;
>> t = ( 0 : 0.001 : 20 ).';
>> v = -1 - 0.1*t;
>> lsim( G, v, t )
```



c) With the P controller the output is given by

$$Y(s) = -\frac{3}{(s+1)}KY(s) + \left(-\frac{12}{30(s+1)} + \frac{4}{(s+2)(s+5)}\right)V(s)$$

which means that

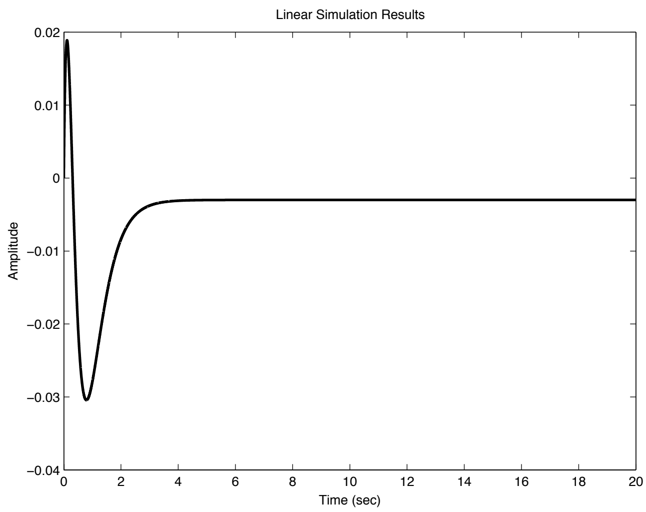
$$Y(s) = \frac{\frac{40(s+1)-4(s+2)(s+5)}{10(s+1)(s+2)(s+5)}}{1+\frac{3K}{s+1}}V(s) = \frac{-0.4s^2+1.2s}{(s+3K+1)(s+2)(s+5)}V(s)$$

Using the same *disturbance*, $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$, the *final value theorem* gives (verify that the system is stable)

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} s\frac{-0.4s^2+1.2s}{(s+3K+1)(s+2)(s+5)}\left(-\frac{1}{s} - \frac{0.1}{s^2}\right)\\ &= \frac{1.2}{(3K+1)\cdot 10}\cdot(-0.1) = -\frac{0.012}{3K+1}\end{aligned}$$

Create the new closed loop system with different values on K.

```
>> K = 1;
>> Gc = minreal( G / ( 1 + K * Gu ) );
>> lsim( Gc, v, t )
```



d) When only a P controller is used we have the following relationship between the *disturbance* and the output

$$Y(s) = -\frac{3}{(s+1)}KY(s) + \frac{4}{(s+2)(s+5)}V(s)$$

which means that

$$Y(s) = \frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}V(s)$$

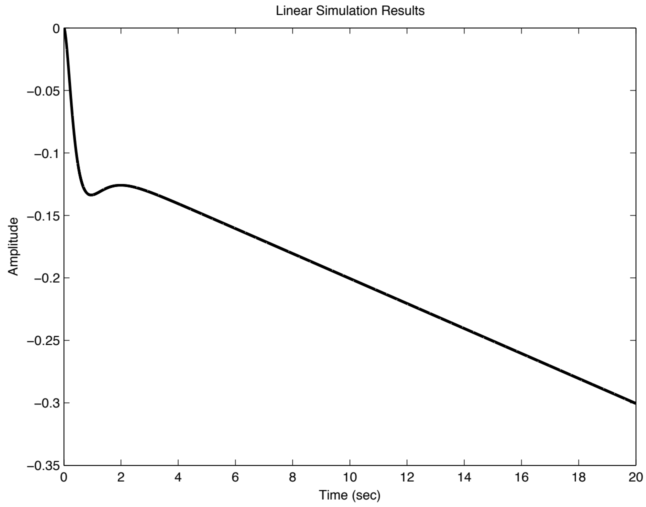
Again using the same *disturbance*, $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$, a careful inspection of $Y(s)$ gives that there is no final value of y, hence the *final value theorem* does not apply.* However, the possibility to simulate the system remains.

*If it is assumed that the final value exists, a contradiction follows since then the *final value theorem* would apply, but give

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} s\frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}\left(-\frac{1}{s} - \frac{0.1}{s^2}\right)\\ &= -\lim_{s\to 0}\frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}\frac{s+0.1}{s} = -\infty\end{aligned}$$

Simulate the output.

```
>> Gc = minreal( Gv / ( 1 + K * Gu ) );
>> lsim( Gc, v, t )
```



Go back

8 State Space Description

8.1 According to Solution 2.1 the differential equation for the motor is

$$\ddot{\theta} + \frac{1}{\tau}\dot{\theta} = Ku$$

where

$$\frac{fR_{\mathrm{a}} + k_a k_v}{JR_{\mathrm{a}}} = \frac{1}{\tau} \quad \frac{k_a}{JR_{\mathrm{a}}} = K$$

Introduce the state variables x_1 and x_2 according to

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

This gives the state space equations

$$\begin{aligned}\dot{x}_1 &= \dot{\theta} = x_2\\ \dot{x}_2 &= \ddot{\theta} = -\frac{1}{\tau}\dot{\theta} + Ku = -\frac{1}{\tau}x_2 + Ku\end{aligned}$$

In matrix form we get

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u\\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

where $x^{\mathrm{T}} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$.

Go back

8.2 We start with the differential equations

$$\ell\ddot{\theta} + g\sin\theta + \ddot{z}\cos\theta = 0$$

The state variables

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

input

$$u = \frac{\ddot{z}}{\ell}$$

and output

$$y = \theta$$

gives the (nonlinear) state space description

$$\begin{aligned}\dot{x}_1 &= x_2 =: f_1(x,u)\\ \dot{x}_2 &= \ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{\ddot{z}}{\ell}\cos\theta = -\omega_0^2\sin x_1 - u\cos x_1 =: f_2(x,u)\end{aligned}$$

where $\omega_0^2 = g/\ell$. We get that

$$\begin{aligned}
\frac{\partial f_1}{\partial x} &= \begin{pmatrix} 0 & 1 \end{pmatrix} \\
\frac{\partial f_1}{\partial u} &= 0 \\
\frac{\partial f_2}{\partial x} &= \begin{pmatrix} -\omega_0^2 \cos x_1 + u \sin x_1 & 0 \end{pmatrix} \\
\frac{\partial f_2}{\partial u} &= -\cos x_1
\end{aligned}$$

Introduce $x_{1\Delta} = x_1 - \pi$, $x_{2\Delta} = x_2$, $u_\Delta = u$, and $y_\Delta = y - \pi$. Linearization around $x_1 = \pi$, $x_2 = 0$ and $u = 0$ gives

$$\begin{aligned}
\dot{x}_{1\Delta} &= x_{2\Delta} \\
\dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\
y_\Delta &= x_{1\Delta}
\end{aligned}$$

Go back

8.3 Introduce the state variables

$$x_1 = y \quad x_2 = \theta \quad x_3 = z$$

According to the figure, the variables are related as

$$\begin{aligned}
X_1(s) = Y(s) &= \frac{1}{s}(M_l(s) + K_2 X_2(s)) \\
X_2(s) = \theta(s) &= \frac{1}{s}(X_3(s) - X_1(s)) \\
X_3(s) = Z(s) &= \frac{1}{s}(K_1 I(s) - K_2 X_2(s))
\end{aligned}$$

Inverse Laplace transformation gives, in the time domain,

$$\begin{aligned}
\dot{x}_1(t) &= K_2 x_2(t) + M_l(t) \\
\dot{x}_2(t) &= -x_1(t) + x_3(t) \\
\dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t)
\end{aligned}$$

In matrix notation this becomes

$$\begin{aligned}
\dot{x}(t) &= \begin{pmatrix} 0 & K_2 & 0 \\ -1 & 0 & 1 \\ 0 & -K_2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} i(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} M_l(t) \\
y(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t)
\end{aligned}$$

Go back

8.4 a) One way is to use a standard form (as supplied in the answer). One can also create a realization manually.

Partial fraction expansion of

$$Y(s) = \frac{2s+3}{s^2+5s+6} U(s)$$

gives

$$Y(s) = -\frac{1}{s+2}U(s) + \frac{3}{s+3}U(s)$$

Introducing the state variables

$$X_1(s) = -\frac{1}{s+2}U(s) \quad X_2(s) = \frac{3}{s+3}U(s)$$

gives

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) - u(t)\\ \dot{x}_2(t) &= -3x_2(t) + 3u(t)\end{aligned}$$

in the time domain. Furthermore, we have

$$y(t) = x_1(t) + x_2(t)$$

In matrix form

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix}-2 & 0\\ 0 & -3\end{pmatrix}x(t) + \begin{pmatrix}-1\\ 3\end{pmatrix}u(t)\\ y(t) &= \begin{pmatrix}1 & 1\end{pmatrix}x(t)\end{aligned}$$

b) One way is to use a standard form (as supplied in the answer). One can also create a realization manually by noting that there are no derivatives on the inputs which means the derivatives of y can be used as states

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) + 6\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 11\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 6y(t) = 6u(t)$$

the state variables

$$x_1(t) = y \quad x_2(t) = \dot{y} \quad x_3(t) = \ddot{y}$$

gives

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= \frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) = -6\ddot{y}(t) - 11\dot{y}(t) - 6y(t) + 6u(t)\\ &= -6x_3(t) - 11x_2(t) - 6x_1(t) + 6u(t)\end{aligned}$$

In matrix form we get

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{pmatrix}x(t) + \begin{pmatrix}0\\ 0\\ 6\end{pmatrix}u(t)\\ y(t) &= \begin{pmatrix}1 & 0 & 0\end{pmatrix}x(t)\end{aligned}$$

c) One way is to use a standard form (as supplied in the answer). One can also create a realization manually although it is far more complicated than before. With

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 5\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = 4\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) + \frac{\mathrm{d}}{\mathrm{d}t}u(t) + 2u(t)$$

If we introduce $x_1(t) = y(t)$ in the equation and collect all terms without differentiation on the right hand side we get

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}x_1(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(t) + 5\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) - 4\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) - \frac{\mathrm{d}}{\mathrm{d}t}u(t) = -3x_1(t) + 2u(t)$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + 5x_1(t) - 4\frac{\mathrm{d}}{\mathrm{d}t}u(t) - u(t)\right) = -3x_1(t) + 2u(t)$$

Now introduce the expression within the parenthesis as a new state variable

$$x_2(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + 5x_1(t) - 4\frac{\mathrm{d}}{\mathrm{d}t}u(t) - u(t)$$

that is

$$\dot{x}_2(t) = -3x_1(t) + 2u(t) \tag{8.1}$$

Repeating this procedure yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + x_1(t) - 4u(t)) = x_2(t) - 5x_1(t) + u(t) \tag{8.2}$$

and we can introduce

$$x_3(t) = \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + x_1(t) - 4u(t)$$

that is

$$\dot{x}_1(t) = x_3(t) - x_1(t) + 4u(t) \tag{8.3}$$

Equation (8.1), (8.2), and (8.3) define the state space equations

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -1 & 0 & 1 \\ -3 & 0 & 0 \\ -5 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t)\end{aligned}$$

Go back

8.5 The impulse response

$$g(t) = 2e^{-t} + 3e^{-4t}$$

gives the *transfer function*

$$G(s) = \frac{2}{s+1} + \frac{3}{s+4}$$

The output can then be written

$$Y(s) = \underbrace{\frac{2}{s+1}U(s)}_{X_1(s)} + \underbrace{\frac{3}{s+4}U(s)}_{X_2(s)}$$

Defining the state variables as above gives

$$\begin{aligned} sX_1(s) + X_1(s) &= 2U(s) \\ sX_2(s) + 4X_2(s) &= 3U(s) \end{aligned}$$

which in time domain can be written as

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + 2u(t) \\ \dot{x}_2(t) &= -4x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t) \end{aligned}$$

Go back

8.6 The *transfer function* is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} s+2 & -1 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{(s+2)(s+3)} \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} s+3 & 1 \\ 0 & s+2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{s}{(s+2)(s+3)} \end{aligned}$$

Go back

8.7 a)

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \quad \Rightarrow \quad x_1 = 1 - e^{-t} \\ \dot{x}_2 &= 2x_2 + u \quad \Rightarrow \quad x_2 = 0.5(e^{2t} - 1) \end{aligned}$$

b) The system is not asymptotically stable since $x_2 \to \infty$ as $t \to \infty$, but input-output stable because the *transfer function* has its pole in the complex left hand plane.

c)

$$\mathcal{S} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad \det \mathcal{S} = 3$$

The system is *controllable*.

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \qquad \det \mathcal{O} = 0$$

The system is not observable. $\mathcal{O}x = 0$ has solutions

$$x = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

This implies that the second component of the state vector cannot be seen in the output.

d) Because the second component of the state vector has *unconstrained* growth and this is not reflected in the output, the system will finally collapse.

Go back

8.8

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s-1 & 1 \\ -2 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{s+1}{(s-1)^2 + 2} \end{aligned}$$

This gives poles in $1 \pm \mathrm{i}\sqrt{2}$ and zeros in -1.

Go back

8.9 a) For pendulum 1 we have

$$\ddot{z}\cos(\phi_1) + \alpha\ddot{\phi}_1 = \sin(\phi_1)$$

and for pendulum 2

$$\ddot{z}\cos(\phi_2) + \ddot{\phi}_2 = \sin(\phi_2)$$

Linearization gives

$$\begin{aligned}\ddot{z} + \alpha\ddot{\phi}_1 &= \phi_1\\ \ddot{z} + \ddot{\phi}_2 &= \phi_2\end{aligned}$$

Consider $\ddot{z}$ as an input to the system (the acceleration of the trolley $\sim$ the force applied to the system). Introduce the state variables

$$x_1 = \phi_1 \quad x_2 = \dot{\phi}_1 \quad x_3 = \phi_2 \quad x_4 = \dot{\phi}_2$$

This gives the state space equations

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{\alpha}x_1 - \frac{u}{\alpha}\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= x_3 - u\end{aligned}$$

In matrix form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1/\alpha & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} + \begin{pmatrix} 0\\ -1/\alpha\\ 0\\ -1\end{pmatrix} u$$

b) The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 0 & -1/\alpha & 0 & -1/\alpha^2\\ -1/\alpha & 0 & -1/\alpha^2 & 0\\ 0 & -1 & 0 & -1\\ -1 & 0 & -1 & 0\end{pmatrix} \qquad \det \mathcal{S} = \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2$$

Thus, the system is controllable except for the case $\alpha = 1$, that is, when the two pendulums have the same lengths. If the pendulums have different lengths they react differently to the input, but if they have the same length there is no possibility to act upon them separately using the input.

Go back

8.10 The figure gives

$$X_1(s) = \frac{1}{(s+1)}U(s) \quad \Rightarrow \quad sX_1(s) = -X_1(s) + U(s)$$

and

$$X_2(s) = \frac{1}{(s+3)}(U(s) + X_1(s)) \quad \Rightarrow \quad sX_2(s) = -3X_2(s) + U(s) + X_1(s)$$

Inverse Laplace transformation gives

$$\begin{aligned}\dot{x}_1 &= -x_1 + u\\ \dot{x}_2 &= -3x_2 + x_1 + u\end{aligned}$$

In matrix form this becomes

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0\\ 1 & -3\end{pmatrix} x + \begin{pmatrix} 1\\ 1\end{pmatrix} u\\ y &= \begin{pmatrix} 1 & 1\end{pmatrix} x\end{aligned}$$

Go back

8.11 a) A stationary point fulfills $f(x_0,u_0)=0,\ \ y=h(x_0,u_0)$.

$$
\begin{aligned}
f_1(x,u) &= x_2 \Rightarrow f_1(x_0,u_0) = x_{20} = 0\\
f_2(x,u) &= -\frac{ku(t)}{mx_1^2(t)} + g \Rightarrow f_2(x_0,u_0) = -\frac{ku_0}{mx_{10}^2} + g = 0\\
y &= x_1 \Rightarrow y_0 = x_{10}
\end{aligned}
$$

For any stationary $y = x_{10} \neq 0$, we can rewrite the second row as

$$f_2(x_0,u_0) = -\frac{ku_0}{mx_{10}^2} + g = 0 \iff u_0 = g\frac{mx_{10}^2}{k}$$

This means that any position (any $x_{10} \neq 0$) can be a stationary point, with the corresponding input $u_0 = g\frac{mx_{10}^2}{k}$. The vertical velocity x_{20} has to be zero.

b) Use

$$
\begin{array}{ll}
A = f_x(x_0,u_0) & B = f_u(x_0,u_0)\\
C = h_x(x_0,u_0) & D = h_u(x_0,u_0)
\end{array}
$$

where

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0,u_0) & \frac{\partial f_1}{\partial x_2}(x_0,u_0)\\ \frac{\partial f_2}{\partial x_1}(x_0,u_0) & \frac{\partial f_2}{\partial x_2}(x_0,u_0)\end{pmatrix} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u}(x_0,u_0)\\ \frac{\partial f_2}{\partial u}(x_0,u_0)\end{pmatrix}$$

Differentiation gives the A matrix elements

$$
\begin{aligned}
f_1(x,u) = x_2 &\Rightarrow \frac{\partial f_1}{\partial x_1}(x_0,u_0) = 0, \text{ and } \frac{\partial f_1}{\partial x_2}(x_0,u_0) = 1\\
f_2(x,u) = -\frac{ku(t)}{mx_1^2(t)} + g &\Rightarrow \frac{\partial f_2}{\partial x_1} = \frac{2ku}{mx_1^3} \Rightarrow\\
\frac{\partial f_2}{\partial x_1}(x_0,u_0) &= \frac{2kgmx_{10}^2}{kmx_{10}^3} = \frac{2g}{x_{10}}, \text{ and } \frac{\partial f_2}{\partial x_2} = 0
\end{aligned}
$$

and the B matrix

$$
\begin{aligned}
\frac{\partial f_1}{\partial u} &= 0\\
\frac{\partial f_2}{\partial u} &= -\frac{k}{mx_1^2} \Rightarrow \frac{\partial f_2}{\partial u}(x_0,u_0) = -\frac{k}{mx_{10}^2}.
\end{aligned}
$$

We thus have the matrices A, B and C (already linear)

$$
A = \begin{pmatrix} 0 & 1\\ \frac{2g}{x_{10}} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0\\ -\frac{k}{mx_{10}^2}\end{pmatrix}
$$
$$
C = \begin{pmatrix} 1 & 0 \end{pmatrix}
$$

and the linearized *state space model* is described by

$$
\begin{aligned}
\Delta\dot{x} &= A\Delta x + B\Delta u\\
\Delta y &= C\Delta x.
\end{aligned}
$$

Go back

8.12 The first part of the block diagram gives

$$Q(s) = \frac{k_V}{s\tau_V + 1}U(s)$$

Multiplying both sides with the denominator polynomial gives

$$(s\tau_V + 1)Q(s) = k_V U(s)$$

which gives

$$\tau_V sQ(s) + Q(s) = k_V U(s)$$

Inverse Laplace transform and some rewriting gives

$$\dot{q}(t) = -\frac{1}{\tau_V}q(t) + \frac{k_V}{\tau_V}u(t)$$

and hence

$$\dot{x}_2(t) = -\frac{1}{\tau_V}x_2(t) + \frac{k_V}{\tau_V}u(t)$$

In a similar way the second part of the diagram gives

$$Y(s) = \frac{k_D}{s\tau_D + 1}(Q(s) + V(s))$$

which leads to the state equation

$$\dot{x}_1(t) = -\frac{1}{\tau_D}x_1(t) + \frac{k_D}{\tau_D}x_2(t) + \frac{k_D}{\tau_D}v(t)$$

Go back

8.13 a) Steady state is when the object moves with constant velocity, i.e. when $\dot{x}(t) = 0$. This gives that the steady state velocity is given by

$$0 = -cx_0^2 + u_0$$

which gives

$$x_0 = \sqrt{\frac{u_0}{c}}$$

The system is non-linear since, for example, doubling the force does not lead to that the velocity is doubled. Instead, in order to double the velocity the force has to be made four times larger.

b) Using the general form for non-linear systems

$$\dot{x}(t) = f(x(t), u(t))$$

we have

$$f(x(t), u(t)) = -\frac{c}{m}x^2(t) + \frac{1}{m}u(t)$$

This implies that

$$f_x' = -2\frac{c}{m}x(t) \qquad f_u' = \frac{1}{m}$$

Evaluating the partial derivatives in the stationary point and using the variables $\Delta x(t) = x(t) - x_0$ and $\Delta u(t) = u(t) - u_0$ gives

$$\Delta\dot{x}(t) = -2\frac{c}{m}x_0\Delta x(t) + \frac{1}{m}\Delta u(t)$$

c) The pole of the linearized system is $-2\frac{c}{m}x_0$ and it describes how the velocity of the object behaves around its steady state value. The expression says that the higher the steady state velocity is the faster will the velocity return to its steady state value if it is perturbed by for example some disturbance. This is due to the fact that the air drag is proportional to the square of the velocity.

Go back

9 State Feedback

9.1 a) The control law

$$u = -Lx + r$$

gives the closed loop system

$$\dot{x} = (A - BL)x + Br$$

and the poles of the closed loop system are given by the eigenvalues of $A - BL$.

$$A - BL = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \end{pmatrix} = \begin{pmatrix} -2 - l_1 & -1 - l_2 \\ 1 & 0 \end{pmatrix}$$

The characteristic equation is given by

$$\det(sI - A + BL) = s^2 + (2 + l_1)s + 1 + l_2 = 0$$

Poles in $\{-3, -5\}$ implies that we will have the equation

$$(s + 3)(s + 5) = s^2 + 8s + 15 = 0$$

Identification of the coefficients gives

$$l_1 = 6 \quad l_2 = 14$$

This gives the control law

$$u = -6x_1 - 14x_2 + r$$

Similarly, poles in $\{-10, -15\}$ gives

$$l_1 = 23 \quad l_2 = 149$$

corresponding to the control law

$$u = -23x_1 - 149x_2 + r$$

One observes that the coefficients in the control law increase when the poles are placed further into the left half plane. In a physical system, this means that larger forces/*voltage*s/flow/... are required to realize the control law.

b) We have $G_c(s) = C(sI - (A - BL))^{-1}Bl_0$ which evaluates to $\frac{l_0}{s^2+8s+15}$ for the first case. For this system to have *static gain* 1 (which is required for the output to converge to the constant reference) we must have $l_0 = 15$.

Go back

9.2 The *feedback* $u = -Lx + l_0 r$ gives the closed loop system

$$\dot{x} = (A - BL)x + Bl_0 r$$

with characteristic equation

$$s^2 + (1 + l_1 + l_2)s + l_1 = 0$$

Poles in $\{-2, -3\}$ implies that we will have the equation

$$(s + 3)(s + 2) = s^2 + 5s + 6 = 0$$

Identification of the coefficients gives

$$l_1 = 6 \quad l_2 = -2$$

and the control law becomes

$$u = -6x_1 + 2x_2 + l_0 r$$

Introduce the observer

$$\dot{\hat{x}}(t) = A\hat{x} + Bu(t) + K(y(t) - C\hat{x}(t))$$

It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system, for example, in -4. The characteristic equation of the observer is

$$s^2 + (1 + k_1 - k_2)s + k_1 = 0$$

and poles in -4 corresponds to the equation

$$s^2 + 8s + 16 = 0$$

Identification of coefficients gives

$$k_1 = 16 \quad k_2 = 9$$

The complete system, that is, the closed loop system with reconstructed states, will have poles in $\{-2, -3\}$, and the observer will have poles in $\{-4, -4\}$.

Go back

9.3 a) Introduce the state variables

$$x_1 = \dot{z} \quad x_2 = \theta \quad x_3 = \dot{\theta}$$

The figure gives the state equations

$$\begin{aligned} X_1(s) &= \frac{1}{s}K_2 X_2(s) \\ X_2(s) &= \frac{1}{s}X_3(s) \\ X_3(s) &= \frac{1}{s}K_1 U(s) \end{aligned}$$

Inverse Laplace transformation gives

$$\dot{x}_1(t) = K_2 x_2(t) \quad \dot{x}_2(t) = x_3(t) \quad \dot{x}_3(t) = K_1 u(t)$$

In matrix form we get

$$\dot{x}(t) = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u(t)$$

b) Since it is assumed that all states are measurable we apply a state feedback

$$u = -Lx + y_{\text{ref}}$$

which gives the closed loop system

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

where

$$A - BL = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ -K_1 l_1 & -K_1 l_2 & -K_1 l_3 \end{pmatrix}$$

The characteristic equation

$$\det(sI - A + BL) = s^3 + K_1 l_3 s^2 + K_1 l_2 s + K_2 K_1 l_1 = 0$$

All three poles in -0.5 implies that we will have the equation

$$(s + 0.5)^3 = s^3 + 1.5s^2 + 0.75s + 0.125 = 0$$

Identification of the coefficients gives

$$l_1 = \frac{1}{8K_1 K_2} \quad l_2 = \frac{3}{4K_1} \quad l_3 = \frac{3}{2K_1}$$

c) If only x_1 is measurable we have

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

Employ the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

The characteristic equation is

$$\det(sI - A + KC) = s^3 + k_1 s^2 + k_2 K_2 s + k_3 K_2 = 0$$

To get a similar behavior as in a), the poles of the observer are placed to the left of the poles of the closed loop system, for example, in -2. This pole placement corresponds to the equation

$$s^3 + 6s^2 + 12s + 8 = 0$$

Identification of the coefficients gives

$$k_1 = 6 \quad k_2 = 12/K_2 \quad k_3 = 8/K_2$$

Go back

9.4 a) Introduce the state variables

$$x_1 = \theta \quad x_2 = \omega$$

We have $\dot{x}_1 = \dot{\theta} = \omega = x_2$ and $\dot{x}_2 = \dot{\omega} = -\frac{1}{\tau}\omega + c_1 u + c_2 f = -\frac{1}{\tau}x_2 + c_1 u + c_2 f$. This gives the state-space model in matrix representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} f$$

b) The feedback

$$u = -l_1\theta - l_2\omega + l_0\theta_{\text{ref}} = -Lx + l_0\theta_{\text{ref}}$$

gives

$$A - BL = \begin{pmatrix} 0 & 1 \\ -c_1 l_1 & -(c_1 l_2 + 1/\tau) \end{pmatrix}$$

The characteristic equation

$$\det(sI - A + BL) = s^2 + (l_2 c_1 + \frac{1}{\tau})s + c_1 l_1 = 0$$

Poles in $\frac{1}{\tau}(-1 \pm \mathrm{i})$ corresponds to

$$(s + \frac{1-\mathrm{i}}{\tau})(s + \frac{1+\mathrm{i}}{\tau}) = s^2 + \frac{2}{\tau}s + \frac{2}{\tau^2} = 0$$

Identification of the coefficients gives

$$l_1 = \frac{2}{c_1\tau^2} \quad l_2 = \frac{1}{\tau c_1}$$

This gives the closed loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2/\tau^2 & -2/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} l_0\theta_{\mathrm{ref}} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} f$$

To derive l_0 we could derive the transfer function from θ_{ref} to θ and ensure the *static gain* is 1. Alternatively, at steady state, that is, when $\dot{x}_1 = \dot{x}_2 = 0$, we should have $\theta = \theta_{\mathrm{ref}}$ when $f = 0$. $\dot{x}_1 = 0$ implies that $x_2 = 0$, and $\dot{x}_2 = 0$ then gives

$$\frac{-2}{\tau^2}x_1 + c_1 l_0 \theta_{\mathrm{ref}} = 0$$

so that

$$l_0 = \frac{2}{c_1\tau^2}$$

The resulting control law becomes

$$u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\mathrm{ref}}$$

c) Introduce the integrated control error as an extra state:

$$\dot{x}_3 = \theta_{\mathrm{ref}} - \theta$$

The new state equations become

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1/\tau & 0 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} f + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \theta_{\mathrm{ref}}$$

Using the *feedback* law

$$u = -l_1\theta - l_2\omega - l_3 x_3$$

we get the state derivative term

$$\begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} u = \begin{pmatrix} 0 & 0 & 0 \\ -c_1 l_1 & -c_1 l_2 & -c_1 l_3 \\ 0 & 0 & 0 \end{pmatrix} x$$

and hence the closed loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ -c_1 l_1 & -1/\tau - c_1 l_2 & -c_1 l_3 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} f + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \theta_{\mathrm{ref}}$$

The poles of the closed loop system are the eigenvalues of the "A" matrix, that is, they are given by the characteristic equation

$$\det \begin{pmatrix} -\lambda & 1 & 0 \\ -c_1 l_1 & -1/\tau - c_1 l_2 - \lambda & -c_1 l_3 \\ -1 & 0 & -\lambda \end{pmatrix} = 0$$

Writing out and changing sign yields

$$\lambda^3 + (c_1 l_2 + \frac{1}{\tau})\lambda^2 + c_1 l_1 \lambda - c_1 l_3 = 0$$

Poles in $\{ \frac{1}{\tau}(-1 \pm \mathrm{i}), \frac{1}{\tau}(-2) \}$ correspond to the equation

$$\lambda^3 + \frac{4}{\tau}\lambda^2 + \frac{6}{\tau^2}\lambda + \frac{4}{\tau^3} = 0$$

where the coefficients may be identified as:

$$l_1 = \frac{6}{c_1 \tau^2} \quad l_2 = \frac{3}{c_1 \tau} \quad l_3 = -\frac{4}{c_1 \tau^3}$$

The resulting control law becomes (note that the *static gain* is 1 by construction, so there is no "l_0" in this controller)

$$\dot{x}_3 = \theta_{\text{ref}} - \theta$$
$$u = -\frac{6}{c_1 \tau^2}\theta - \frac{3}{c_1 \tau}\omega + \frac{4}{c_1 \tau^3} x_3$$

Go back

9.5 The system has the observability matrix

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

that is, $\det \mathcal{O} \neq 0$. The system is observable and thus the poles of the observer may be placed *arbitrarily*.

Go back

9.6 The system is described in matrix form by

$$\dot{x}(t) = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t)$$

a) Arbitrary values of the states can be reached if the system is *controllable*. The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

and since $\det \mathcal{S} = 1$ the system is *controllable* and an *arbitrary* temperature profile can be obtained.

b) How the state decays depends on the poles of the closed loop system. Poles in -3 will yield the desired result. The closed loop system,

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

$$A - BL = \begin{pmatrix} -2 - l_1 & 1 - l_2 & -l_3 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

has the characteristic equation

$$s^3 + (6 + l_1)s^2 + (10 + 4l_1 + l_2)s + 4 + 3l_1 + 2l_2 + l_3 = 0$$

Poles in -3 implies that this coincide with the equation

$$(s + 3)^3 = s^3 + 9s^2 + 27s + 27 = 0$$

Identification of the coefficients gives

$$l_1 = 3 \quad l_2 = 5 \quad l_3 = 4$$

Thus, the control law is given by

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\text{ref}}$$

c) Check when the system is observable. The sensor at x_1 corresponds to $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix} \qquad \det \mathcal{O} = 1$$

The sensor at x_2 corresponds to $C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ -4 & 6 & -4 \end{pmatrix} \qquad \det \mathcal{O} = 0$$

The sensor at x_3 corresponds to $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 5 \end{pmatrix} \qquad \det \mathcal{O} = -1$$

The system is hence observable when the sensor is placed at x_1 or x_3, but not with the sensor placed at x_2. That is, the specifications may be fulfilled with the sensor placed at x_1 or x_3. If the sensor is placed at x_1, the characteristic equation of the observer is given by

$$s^3 + (6 + k_1)s^2 + (10 + 4k_1 + k_2)s + 4 + 3k_1 + 2k_2 + k_3 = 0$$

Placing the poles in -4 (which is somewhat faster than the nominal closed loop system) corresponds to the equation

$$(s + 4)^3 = s^3 + 12s^2 + 48s + 64 = 0$$

Identification of coefficients gives

$$k_1 = 6 \quad k_2 = 14 \quad k_3 = 14$$

Go back

9.7 a) Enter the *transfer function* and generate the *state space model*.

```
>> s = tf( 's' );
>> G = ss( 1 / ( s * ( s + 1 ) ) )
a =
        x1  x2
   x1   -1  -0
   x2    1   0

b =
        u1
   x1    1
   x2    0

c =
        x1  x2
   y1    0   1

d =
        u1
   y1    0

Continuous-time model.
```

We hence have the *state space model*

$$
\begin{aligned}
\dot{x}(t) &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)
\end{aligned}
$$

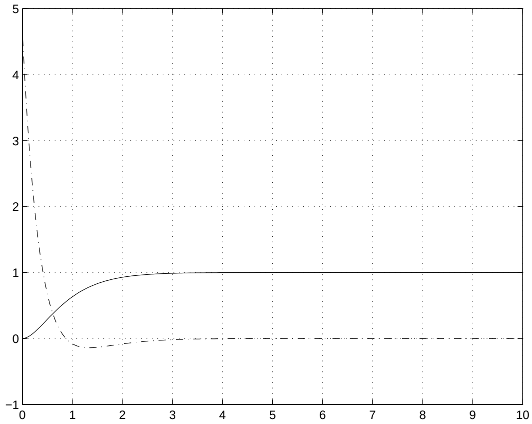
From the last equation we have $x_2(t) = y(t)$, that is, x_2 is the motor angle. From the first equation we have $\dot{x}_2(t) = x_1(t)$, that is, x_1 is the angular velocity.

b) Compute *feedback* gains. The gain l_0 is computed by constructing a system with $l_0 = 1$ first, and then correcting l_0 by the inverse of that system's *static gain*.

```
>> L = place( G.a, G.b, [ -2.2 -2.1 ] );
>> Gctemp = ss( G.a - G.b * L, G.b, G.c, 0 );
>> l_0 = 1 / dcgain( Gctemp);
>> G_r_to_y = ss(G.a - G.b * L, G.b*l_0, G.c, 0);
```

Calculate the *step response* and the control signal. To see the control signal we define a second system which outputs $-Lx + l_0 r$ instead of $Cx + 0r$.

```
>> G_r_to_u = ss(G.a - G.b * L, G.b*l_0, -L, l_0);
>> step(G_r_to_y,G_r_to_u);
>> grid
```

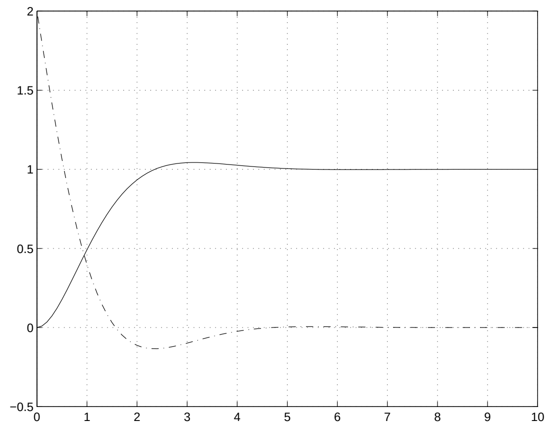


Compute a new *feedback*. This time, we compute the gain l_0 manually by using the formula for the *static gain* of the closed loop system system with $l_0 = 1$ (put $s = 0$ in the generic expression for the *transfer function*).

```
>> L = place( G.a, G.b, [ -1+i -1-i ] );
>> l_0 = 1/( G.c*inv(-G.a+G.b*L)*G.b );
>> G_r_to_y = ss(G.a-G.b*L,G.b*l_0,G.c, 0 );
>> G_r_to_u = ss(G.a-G.b*L,G.b*l_0,-L, l_0 );
```

Calculate the *step response* and the corresponding control signal. Plot the result.

```
>> step(G_r_to_y,G_r_to_u);
>> grid
```



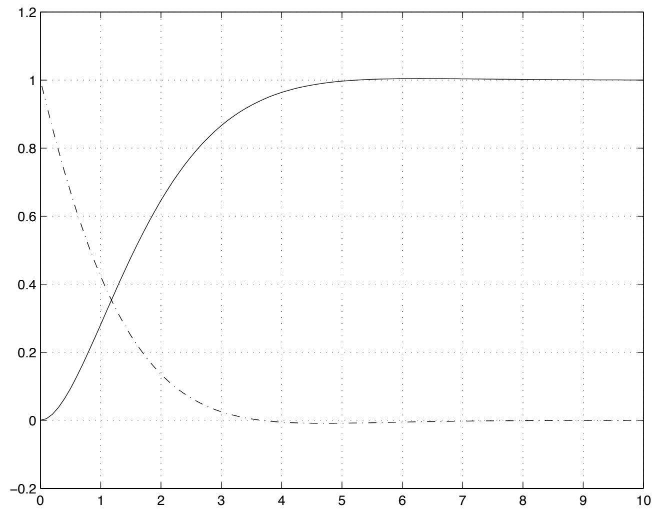
The *step responses* have approximately the same rise and *settling times*. By choosing the closed loop poles complex, and hence allowing a small *overshoot* in the step response, we have however reduced the maximum value of the input signal significantly.

c) Case (i): Compute the *feedback* gain L, l_0, and the closed loop system.

```
>> L = lqr( G.a, G.b, diag([ 0 1 ]), 1);
>> l_0 = 1 / ...
     ( G.c * inv( -G.a + G.b*L ) * G.b);
>> G_r_to_y = ss(G.a-G.b*L,G.b*l_0,G.c, 0);
>> G_r_to_u = ss(G.a-G.b*L,G.b*l_0,-L, l_0);
```

Simulate the system and plot the result.

```
>> step(G_r_to_y,G_r_to_u);
>> grid
```



Compute the closed loop poles. This time, via the eigenvalues of the "A" matrix.

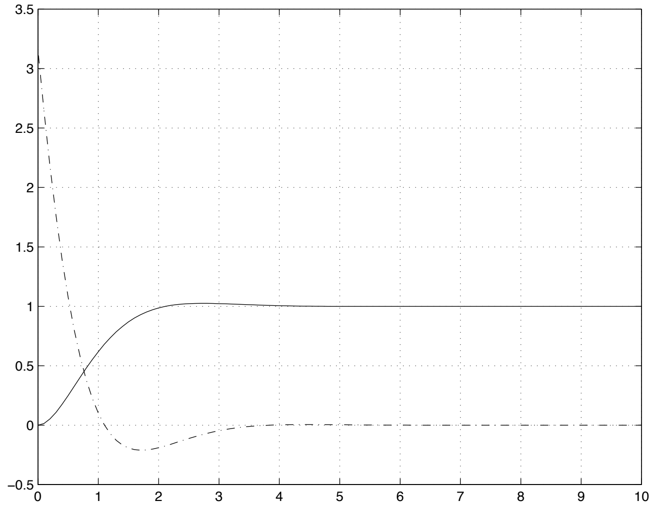
```
>> eig( G_r_to_y.a )
ans =
  -0.8660 + 0.5000i
  -0.8660 - 0.5000i
```

Case (ii): Repeat, this time with larger weight on the motor angle.

```
>> L = lqr( G.a, G.b, diag([ 0 10 ]), 1 );
>> l_0 = 1 / ...
( G.c * inv( -G.a + G.b*L ) * G.b );
>> G_r_to_y = ss(G.a-G.b*L,G.b*l_0,G.c, 0 );
>> G_r_to_u = ss(G.a-G.b*L,G.b*l_0,-L, l_0 );
```

Simulate the system and plot the result. The *step response* is now significantly faster.

```
>> step(G_r_to_y,G_r_to_u);
>> grid
```



Compute the closed loop poles. This time using a dedicated command from the toolbox instead of the standard `eig` command. The poles are now further away from the origin and the relative damping is slightly reduced.

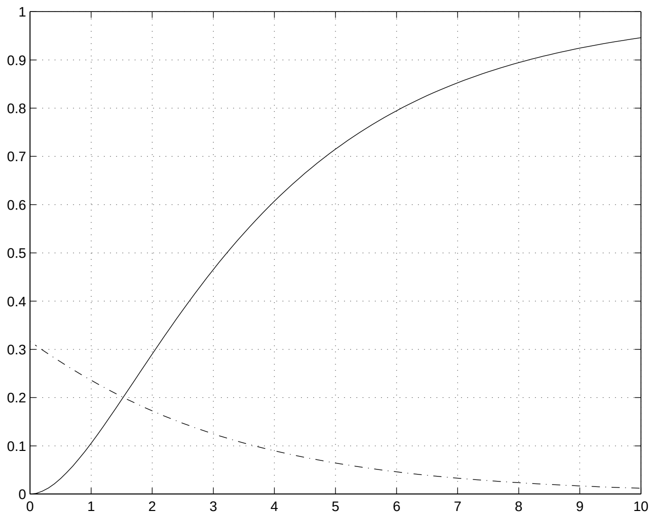
```
>> pole( G_r_to_y )
ans =
  -1.3532 + 1.1537i
  -1.3532 - 1.1537i
```

Case (iii): Repeat, this time with smaller weight on the motor angle.

```
>> L = lqr( G.a, G.b, diag([ 0 0.1 ]), 1 );
>> l_0 = 1 / ...
( G.c * inv( -G.a + G.b*L ) * G.b );
>> G_r_to_y = ss(G.a-G.b*L,G.b*l_0,G.c, 0 );
>> G_r_to_u = ss(G.a-G.b*L,G.b*l_0,-L, l_0 );
```

Simulate the system and plot the result. The *step response* is now much slower.

```
>> step(G_r_to_y,G_r_to_u);
>> grid
```



Compute the closed loop poles. We now get two real closed loop poles, where the pole in -0.34 causes the slow *step response*.

```
>> pole( G_r_to_y )
ans =
    -0.9420
    -0.3357
```

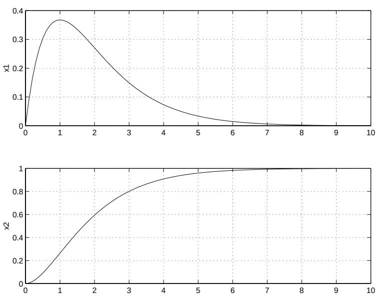
d) If we start from case (ii) and increase the matrix called `R` in the call to `lqr`, the closed loop system gradually becomes slower, since we put an increasing weight on the control signal magnitude. When we reach $R = 10$ we get exactly the same result as for case (i). Since it is the "ratio" between `Q` and `R` that determines the closed loop properties we get the same *feedback* gain if we scale `Q` and `R` by the same scalar.

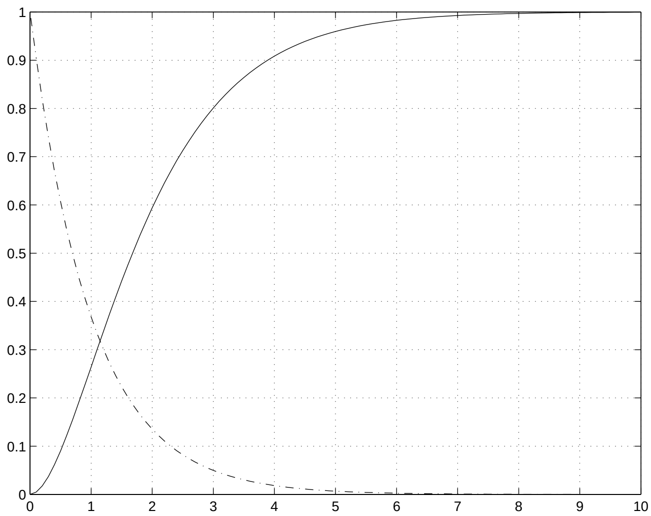
e) Compute *feedback* gains, adjust *static gain*, and compute closed loop system, but this time output all the states instead of y

```
>> L = lqr( G.a, G.b, diag([ 1 1 ]), 1);
>> l_0 = 1 / ...
( G.c * inv( -G.a + G.b*L ) * G.b);
>> G_r_to_x = ss(G.a-G.b*L,G.b*l_0,eye(2), 0);
>> G_r_to_u = ss(G.a-G.b*L,G.b*l_0,-L, l_0);
```

Simulate the system plot the states, x_1 and x_2

```
>> step(G_r_to_x);
>> grid
```





Increasing the weight on the angular velocity forces the motor to move slower, and then also the *step response* becomes slower.

Go back

9.8 Introduce the state variables

$$x_1(t) = q(t) \quad x_2(t) = m(t)$$

This gives the state space description

$$\dot{x}(t) = \begin{pmatrix} -0.05 & 0 \\ 0.05 & -0.02 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

a) The system has the controllability matrix

$$\mathcal{S} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -0.05 \\ 0 & 0.05 \end{pmatrix} \qquad \det \mathcal{S} = 0.05$$

Thus, the system is *controllable*.

b) The control law

$$u(t) = -Lx(t)$$

gives the closed loop system

$$\dot{x}(t) = (A - BL)x(t)$$

and the poles of the closed loop system is given by the eigenvalues of $A - BL$.

$$A - BL = \begin{pmatrix} -0.05 - l_1 & -l_2 \\ 0.05 & -0.02 \end{pmatrix}$$

The characteristic equation is given by

$$\det(sI - A + BL) = s^2 + (0.07 + l_1)s + 0.001 + 0.02l_1 + 0.05l_2 = 0$$

Both poles in -0.1 implies that we shall have the equation

$$(s + 0.1)^2 = s^2 + 0.2s + 0.01 = 0$$

Identification of the coefficients gives

$$l_1 = 0.13 \quad l_2 = 0.128$$

This gives the control law

$$u(t) = -0.13x_1(t) - 0.128x_2(t)$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large *amplification* of the measurement noise the poles of the observer should not be placed too far into the left hand plane.

d) Only $y(t) = x_2(t)$ is measurable. Employ the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

The characteristic equation is

$$\det(sI - A + KC) = s^2 + (0.07 + k_2)s + 0.05k_1 + 0.05k_2 + 0.001$$

Both poles in -0.2 implies that we shall have the equation

$$s^2 + 0.4s + 0.04 = 0$$

Identification of the coefficients gives

$$k_1 = 0.45 \quad k_2 = 0.33$$

Go back

9.9 a) With the given state variables we get

$$\dot{x}_1 = \dot{y}(t) = x_2(t)$$

and

$$\dot{x}_2 = \ddot{y}(t) = u(t)$$

Introducing the state vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

the system can be expressed in state space form as

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

b) Using the proposed feedback the closed loop system becomes

$$\dot{x}(t) = (A - BL)x(t) + B\tilde{r}(t) \qquad y(t) = Cx(t)$$

The poles of the closed loop system are given by

$$\det(sI - (A - BL)) = 0$$

which gives

$$s^2 + l_2 s + l_1 = 0$$

The desired location of the closed loop poles correspond to the characteristic equation

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

Comparing the two equations gives

$$l_1 = \omega_0^2 \quad l_2 = 2\zeta\omega_0$$

Note the intuitive solution. To achieve a faster closed loop (larger ω_0) with unchanged damping (ζ) we increase both the proportional feedback from position (y) and derivative feedback velocity $\dot{y}$. To only increase damping (ζ) we only increase the feedback from the velocity.

c) The transfer function of the closed loop system from $R(s)$ to $Y(s)$ is given by

$$G_C(s) = C(sI - (A - BL))^{-1}Bl_0$$

where we have used that $\tilde{r}(t) = l_0 r(t)$. Inserting the matrices above gives

$$G_C(s) = \frac{l_0}{s^2 + l_2 s + l_1}$$

d) The static gain of the closed loop system is $G_C(0)$, which gives

$$G_C(0) = \frac{l_0}{l_1}$$

When l_0 is chosen such that $G_C(0) = 1$ this gives $l_0 = l_1$. The controller can hence be written

$$u(t) = -l_1 x_1(t) - l_2(t) + l_1 r(t) = l_1(r(t) - y(t) - l_2 x_2(t)$$

In this problem the input $u(t)$ is a force, and for the mass to remain in the desired postion (and there are no other forces acting on the mass) the force has to be zero. In the controller above, with $l_0 = l_1$, the first term is a proportional feedback based on the error $e(t) = r(t) - y(t)$, which zero when $y(t) = r(t)$. The second term is based on the velocity, and this is zero when the mass is at rest.

Go back

9.10 a) Using the function `ss` without semi-colon gives the output below. The matrices of the state space model can be accessed as `G.a, G.b`, etc.

```
>> Gss=ss(G)

Gss =

A =
x1  x2
x1  -1   0
x2   1   0

B =
u1
x1   1
x2   0

C =
x1  x2
y1   0   1

D =
u1
y1   0

Continuous-time state-space model.
```

b) The first set of commands gives Figure 9.10a. The square wave can be seen as a sequence of positive and negative steps.

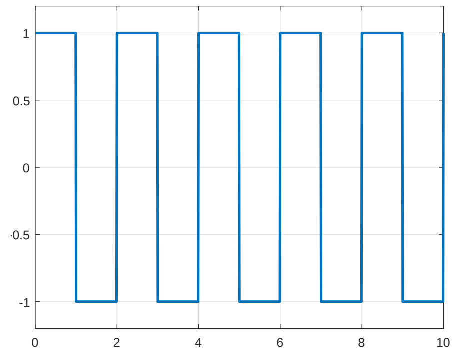


Figure 9.10a. Square wave input.

The second sequence of commands gives Figure 9.10b. Since the input is a sequence of steps and the relationship between input and velocity is a first order systems, the plot of the state $x_1(t)$ becomes a sequence of step responses of a first order system.

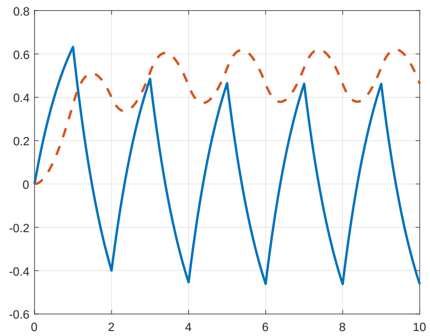


Figure 9.10b. States. Dashed: $x_2(t)$. Solid: $x_1(t)$.

c) Adding the disturbance to the output gives the result in Figure 9.10c.

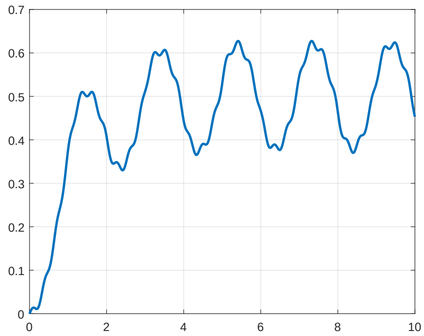


Figure 9.10c. Measured output $y_m(t)$, including the measurement disturbance.

d) The command sequence gives the results in Figure 9.10d. Choosing a smaller time constant in $F(s)$ gives that the measurement disturbance is amplified more and has bigger influence on the estimated velocity. On the other hand, a larger value will cause delay (phase shift) in the estimate.

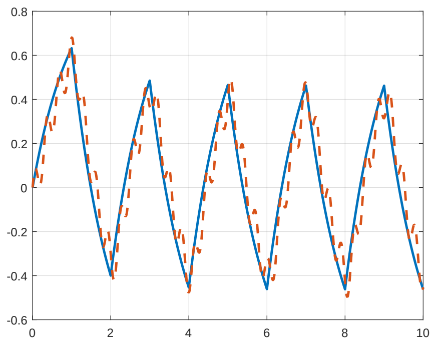


Figure 9.10d. Solid: True velocity. Dashed: Estimated velocity.

e) The sequence of commands gives the result in Figure 9.10e, which shows the estimation error, i.e. the difference between the estimated velocity and the true velocity. As can be seen the error is very small, which means that the estimate is very close to the true signal. One of the key features of using an observer is that we use the input signal $u(t)$ and a model of the system, so that the estimate is generated as a simulation of the model assisted by measured output. Also, in the example here the observer has the same initial values as the true system, which cannot be assumed in practice.

Go back

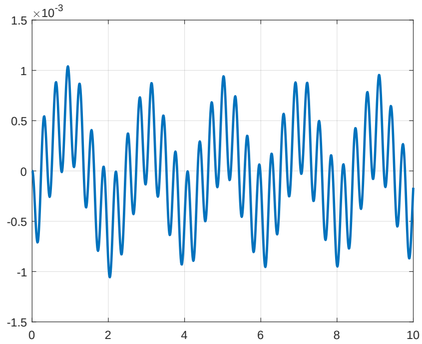


Figure 9.10e. Estimation error $x_1(t) - \hat{x}_1(t)$.

11 Implementation

11.1 Inverse Laplace transformation of

$$U(s) = KN\frac{s+b}{s+bN}E(s)$$

gives the differential equation

$$\dot{u}(t) + bNu(t) = KN\dot{e}(t) + bKNe(t) \tag{11.1}$$

At time $t - T$ we have

$$\dot{u}(t-T) + bNu(t-T) = KN\dot{e}(t-T) + bKNe(t-T) \tag{11.2}$$

By replacing $\dot{u}(t)$ and $\dot{e}(t)$ in (11.1) and (11.2) with $\Delta_t u(t)$ and $\Delta_t e(t)$, respectively, and then adding the equations we get

$$\begin{aligned}&\Delta_t u(t) + \Delta_t u(t-T) + bNu(t) + bNu(t-T)\\ &\qquad = KN\Delta_t e(t) + KN\Delta_t e(t-T) + bKNe(t) + bKNe(t-T)\end{aligned}$$

Tustins formula

$$\frac{1}{2}(\Delta_t u(t) + \Delta_t u(t-T)) = \frac{1}{T}(u(t) - u(t-T))$$

now gives

$$\begin{aligned}&\frac{2}{T}(u(t) - u(t-T)) + bN(u(t) + u(t-T))\\ &\qquad = \frac{2}{T}KN(e(t) - e(t-T)) + bKN(e(t) + e(t-T))\end{aligned}$$

Inserting the numerical values, $K = 2$, $T = 0.1$, $N = 10$ and $b = 0.1$, we get

$$\begin{aligned}&20(u(t) - u(t-T)) + (u(t) + u(t-T))\\ &\qquad = 400(e(t) - e(t-T)) + 2(e(t) + e(t-T))\end{aligned}$$

which gives

$$u(t) = \frac{19}{21}u(t-T) + \frac{402}{21}e(t) - \frac{398}{21}e(t-T)$$

that is

$$u(t) = 0.905u(t-T) + 19.14e(t) - 18.95e(t-T)$$

Go back

Liten reglerteknisk ordlista

This version: 2025-08-21

1 Engelsk-svensk

actuator	ställdon
amplitude	amplitud
attenuation	dämpning
bandwidth	bandbredd
closed loop system	slutet system
control law	styrlag
controllability	styrbarhet
controller	regulator
crossover frequency	skärfrekvens
damping	dämpning
damping ratio	relativ dämpning
disturbance rejection	störningsundertryckning
eigenvalue	egenvärde
feedback	återkoppling
final value theorem	slutvärdesteoremet
feedforward	framkoppling
flow	flöde
gain	förstärkning
gain crossover frequency	(amplitud)skärfrekvens
gain margin	amplitudmarginal
impulse response	impulssvar
initial value	begynnelsevärde
loop gain	kretsförstärkning, öppna systemet
magnitude	amplitud
observability	observerbarhet
observer	observatör
open-loop system	öppet system, kretsförstärkning

overshoot	översläng
peak frequency	resonansfrekvens
peak resonance	resonanstopp
phase crossover frequency	fasskärfrekvens
phase lag	fasretarderande
phase lead	fasavancerande
phase margin	fasmarginal
ramp function	ramp
rank	rang
resonant frequency	resonansfrekvens
rise time	stigtid
root locus (pl. loci)	rotort
sensitivity function	känslighetsfunktion
sensor	givare
settling time	insvängningstid, lösningstid
sinusoidal	sinusformad
state	tillstånd
state feedback	tillståndsåterkoppling
static gain	statisk förstärkning
steady state	stationärt tillstånd
step function	steg
step repsonse	stegsvar
subspace	underrum
time delay	tidsfördröjning
transfer function	överföringsfunktion
unit step	enhetsteg
unstable	instabil

2 Svensk-engelsk

(amplitud)skärfrekvens	gain crossover frequency
aggregerade modeller	lumped models
amplitud	amplitude
amplitud	magnitude
amplitudmarginal	gain margin
bandbredd	bandwidth
begynnelsevärde	initial value
dämpning	damping, attenuation
egenvärde	eigenvalue
enhetsteg	unit step
faltning	convolution
fasavancerande	phase lead
fasmarginal	phase margin
fasretarderande	phase lag
fasskärfrekvens	phase crossover frequency
flöde	flow
framkoppling	feedforward
förstärkning	gain
impulssvar	impulse response
instabil	unstable
insvängningstid, lösningstid	settling time
kretsförstärkning	loop gain, open loop system
känslighetsfunktion	sensitivity function

observatör	observer
observerbarhet	observability
ramp	ramp function
regulator	controller
reglering	control
reglerteknik	automatic control
relativ dämpning	damping ratio
resonansfrekvens	peak frequency
resonansfrekvens	resonant frequency
resonanstopp	peak resonance
rotort	root locus (pl. loci)
sinusformad	sinusoidal
slutet system	closed loop system
skärfrekvens	gain crossover frequency
stationärt tillstånd	steady state
statisk förstärkning	static gain
steg	step function
stegsvar	step repsonse
stigtid	rise time
styrbarhet	controllability
styrlag	control law
ställdon	actuator
störningsundertryckning	disturbance rejection
tidsfördröjning	time delay
tillstånd	state
tillståndsåterkoppling	state feedback
återkoppling	feedback
öppet system	open-loop system, loop gain
överföringsfunktion	transfer function
översläng	overshoot

Introduktion till MATLAB & Control System Toolbox

This version: 2025-08-21

1 Inledning

Denna skrift är en kort inledning till hur MATLAB och Control System Toolbox (CST) används i kurserna i Reglerteknik.

Det absolut viktigaste kommandot i MATLAB är `help`. Tecknen » är bara med för att förtydliga att uttrycket `help tf` är skrivet av oss i MATLABs *kommandoprompt*.

```
>> help help
help Display help text in Command Window.
help NAME displays the help for the functionality specified by NAME,
such as a function, operator symbol, method, class, or toolbox.
NAME can include a partial path.
```

T.ex

```
>> help tf
tf  Construct transfer function or convert to transfer function.

Construction:
SYS = tf(NUM,DEN) creates a continuous-time transfer function SYS with
numerator NUM and denominator DEN....
```

Alternativt kan man använda `doc` som öppnar en browser till hjälpsystemet.

För en komplett nybörjare kan det vara bra att känna till skillnaden mellan klamrar `[]` och paranteser `()` och vad `;` används till.

Paranteser används på precis samma sätt som när vi skriver matematik, dvs för att samla termer och funktionsanrop. Här ser vi även hur vi tilldelar en variabel ett värde med `=`

```
>> a = 5*(2+1)

a =

15
>> b = cos(0)

b =

1
```

Klamrar används när man vill skapa vektorer. Standard så erhålls en radvektor. För förtydligande kan man placera `,` mellan elementen men det är inte nödvändigt. Genom att använda `;` inne i vektorn signalerar man radbyte, dvs vi kan skapa kolonnvektor genom att placera ett `;` efter varje element, och matriser genom att placera `;` efter varje önskad rad.

```
>> a = [1 2 3]

a =

1     2     3

>> a = [1, 2, 3]

a =

1     2     3

>> b = [1; 2; 3]

b =

1
2
3

>> c = [1, 2, 3;4, 5, 6]

c =

1     2     3
4     5     6
```

Ett semikolon signalerar alltså radbyte inne i en matrisdefinition, men om det placeras i slutet på en rad så betyder det istället att MATLAB inte ska skriva ut resultatet. Det betyder också att man kan visa värdet av en variabel genom att bara skriva namnet utan ett semikolon.

```
>> d = [1 2 3;4 5 6];

>> d
d =

1     2     3
4     5     6
```

I koden ovan har vi skrivit in allting direkt i MATLABs kommandotprompt. Det kan man göra när man bara testar lite sporadiskt, men rekommenderat arbetssätt är att man skriver sin kod i MATLABS editor och ser till att spara. Editorn, om den inte är öppen från början, fås upp med `edit`. MATLAB-filer är vanliga textfiler med ändelsen `.m`. Man kan köra alla kommandon i en fil genom att trycka på *Run*-knappen, eller anropa filen från kommandoprompten (det kräver dock att filnamnet inte innehåller mellanslag eller svenska tecken), eller kopiera kod i editorn och klistra in i kommandoprompten, eller markera kod i editorn och högerklicka och välja att evaluera markerad kod.

2 System

I Control System Toolbox finns datastrukturer för att hantera s k *LTI-objects*, dvs linjära tidsinvarianta system, på ett bekvämt sätt. Vi kommer inledningsvis främst att arbeta med system på överföringsfunktionsform, men senare även med system på så kallad tillståndsform. Överföringsfunktion kan skapas på två sätt.

Det smidigaste sättet är att mata in överföringsfunktionen på symbolisk form genom att först skapa ett objekt bestående av symbolen `s`. Därefter kan man t ex addera och multiplicera med denna symbol på samma sätt som görs med Laplace-variabeln s vid handräkning.

OBS, För att få till ˆ nedan måste man i många system skriva *shift* ˆ *mellanslag*.

Betrakta överföringsfunktionen

$$G(s) = \frac{4}{s(s^2 + 2s + 4)} = \frac{4}{s^3 + 2s^2 + 4s}$$

Skapa ett objekt bestående av symbolen `s`. Bilda överföringsfunktionen genom att använda vanliga räkneoperationer.

```
>> s = tf( 's' );
>> G = 4 / ( s * ( s^2 + 2*s + 4 ) )

Transfer function:
        4
-----------------
s^3 + 2 s^2 + 4 s
```

Ett krångligare alternativ är kommandot `tf` med vektorer som beskriver koefficienter i täljarens och nämnarens polynom.

Mata in systemet och ge objektet namnet `G`. Argumenten till funktionen utgörs av radvektorer innehållande täljarens respektive nämnarens koefficienter.

```
>> G = tf( 4, [ 1 2 4 0 ] )

Transfer function:
  4
-----------------
s^3 + 2 s^2 + 4 s
```

LTI-objekt som man skapar kan multipliceras, divideras, adderas och subtraheras på ett rättframt sätt.

Skapa en ny överföringsfunktion `P(s)` genom att seriekoppla $G(s)$ och överföringsfunktionen $H(s) = \frac{1}{s+1}$

```
>> H = 1/(s+1);
>> P = G * H

Transfer function:
            4
-------------------------
s^4 + 3 s^3 + 6 s^2 + 4 s
```

---

3 Poler, nollställen och statisk förstärkning

Poler och nollställen till överföringsfunktioner beräknas med funktionerna `pole` respektive `tzero`. Poler och nollställen kan även ritas (dvs markera positionen i komplexa talplanet) med funktionen `pzmap`. Statisk förstärkning $G(0)$ beräknas `dcgain`.

Beräkna polerna till $G(s)$. Systemet har en reell pol i origo och två komplexa poler vilka returneras i en vektor

```
>> pole( G )

ans =

        0
  -1.0000 + 1.7321i
  -1.0000 - 1.7321i
```

Beräkna nollställena till $G(s)$. Eftersom täljaren i överföringsfunktionen är konstant saknar systemet nollställen.

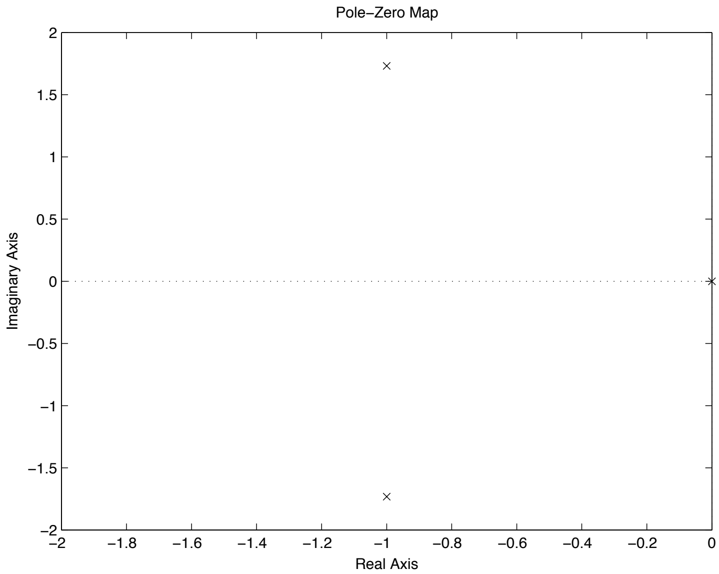
```
>> tzero( G )

ans =

   Empty matrix: 0-by-1
```

Beräkna statisk förstärkning (här oändlig eftersom vi har en pol i origo)

```
>> dcgain( G )

ans =
   Inf
```

Rita in systemets poler och nollställen i det komplexa talplanet. Poler markeras med kryss och nollställen, i de fall de förekommer, markeras med ringar.

```
>> pzmap( G )
>> axis([ -2 0 -2 2 ])
```



4 Återkoppling

I kursen behandlas återkopplade reglersystem, t.ex enligt figur 1 där vi har en referenssignal och en utsignalsstörning.

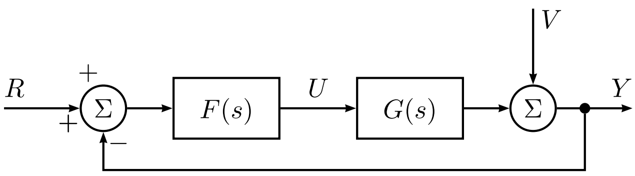


Figure 1. Reglersystem

Med systembeskrivningen

$$Y(s) = G(s)U(s) + V(s)$$

och återkopplingen

$$U(s) = F(s)(R(s) - Y(s))$$

ges det återkopplade systemet av

$$Y(s) = G_c(s)R(s) + S(s)V(s)$$

där

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}$$

och

$$S(s) = \frac{1}{1 + G(s)F(s)}$$

Överföringsfunktionerna för det återkopplade systemet kan enkelt beräknas

---

Generera överföringsfunktionen för en proportionalregulator med förstärkning $K_p = 0.7$ (vi behöver egentligen inte definiera `F` som ett LTI-objekt utan det skulle räcka med `F = 0.7`)

```
>> F = tf( 0.7 )

Transfer function:
 0.7
```

Beräkna överföringsfunktionen för det återkopplade systemet.

```
>> Gc = G*F/(1+G*F)

ans =

  2.8 s^3 + 5.6 s^2 + 11.2 s
---------------------------------------------------
s^6 + 4 s^5 + 12 s^4 + 18.8 s^3 + 21.6 s^2 + 11.2 s
```

Här ska man dock bli misstänksam. Systemet $G(s)$ är av ordning 3 (dvs 3 poler), regulatorn $F(s)$ är av ordning 0 (0 poler), och det slutna systemet är av ordning 6 (dvs 6 poler). Poler motsvarar något fysikaliskt, och här har det helt plötsligt skapats 3 poler, vilket fysikaliskt är omöjligt. Även matematiskt är det omöjligt då vi har multiplicerat och adderat polynom av ordning 3 oh 0, och plötsligt fått ett uttryck av ordning 6. Problemet är att det finns en förkortning mellan nämnare och täljare som MATLAB misslyckats att identifiera.

Tre poler och nollställen är identiska och borde således ha förkortats bort.

```
>> pole(Gc)

ans =

 0.0000 + 0.0000i
 -1.0000 + 1.7321i
 -1.0000 - 1.7321i
 -0.5341 + 1.6491i
 -0.5341 - 1.6491i
 -0.9319 + 0.0000i

>> tzero(Gc)

ans =

 -1.0000 + 1.7321i
 -1.0000 - 1.7321i
 -0.0000 + 0.0000i
```

För att säkerställa att det inte finns oförkortade termer använder man kommandot `minreal`

```
>> Gc = minreal(G*F/(1+G*F))

Gc =

   2.8
-----------------------
s^3 + 2 s^2 + 4 s + 2.8
```

Alternativt kan man använda kommandot `feedback` som gör manipulationerna på ett säkrare sätt vilket gör att man inte erhåller falska poler och nollställen.

```
>> Gc = feedback(G*F,1)

Gc =

  2.8
-----------------------
s^3 + 2 s^2 + 4 s + 2.8
```

`feedback(A,B)` skapar återkopplingen $\frac{A}{1+AB}$. Verifiera att känslighetsfunktionen således kan konstrueras med

```
>> S = feedback(1,G*F)

S =

 s^3 + 2 s^2 + 4 s
-----------------------
s^3 + 2 s^2 + 4 s + 2.8
```

---

5 Simulering

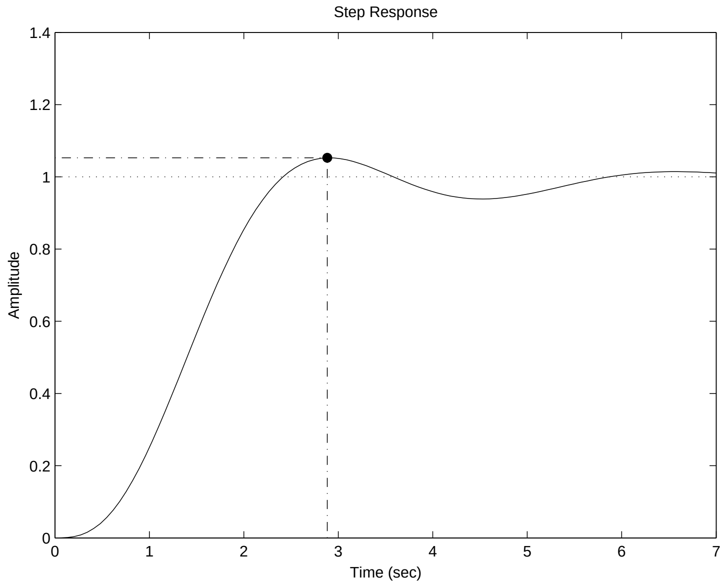
5.1 Stegsvar

Den vanligaste typen av simulering är att beräkna ett systems stegsvar. Detta kan utföras med funktionen `step`, med vilken man både simulerar systemet och ritar dess stegsvar. I likhet med tidigare kan man läsa av enskilda värden i figuren med vänster musknapp och få en meny med olika val med höger knapp. Genom att t ex välja `Peak Response` från `Characteristics` markeras tidpunkt och värde för överslängen. Placera markören över punkten i diagrammet visas tillhörande numeriska värden.

Antag att vi har det återkopplade systemet definierat sedan tidigare och vi vill se responsen för ett enhetssteg.

Beräkna och rita upp det återkopplade systemets stegsvar. Markera stegsvarets översläng.

```
>> step( Gc )
```



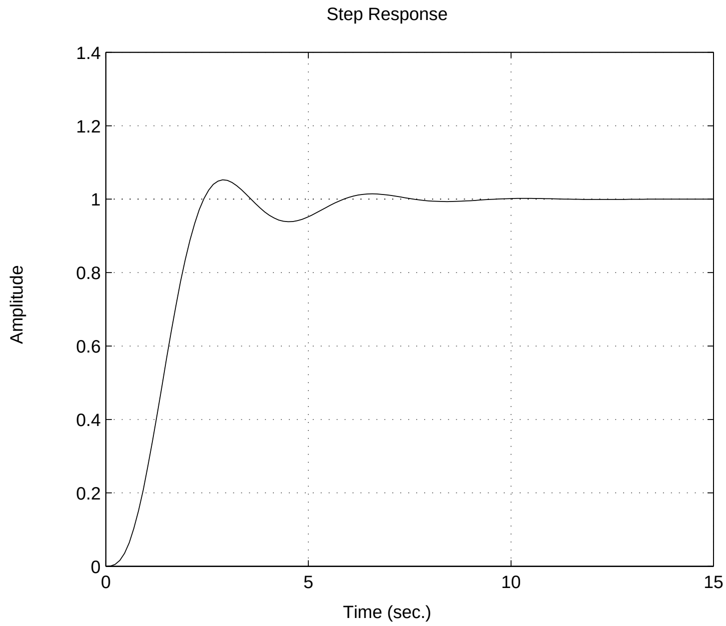
Om man vill göra ett steg med en annan amplitud, t.ex 10, så utnyttjar man enklast linjäritet. Att studera stegsvaret $Y(s) = G_c(s)R(s) = G_c(s)\frac{10}{s}$ är ekvivalent med att studera enhetsstegsvaret $(10G_c(s))\frac{1}{s}$

```
>> step( 10*Gc )
```

I normalfallet väljs simuleringstiden automatiskt, men genom att ange ett extra argument kan man välja simuleringstiden själv.

Beräkna det återkopplade systemets stegsvar under femton sekunder och rita upp resultatet.

```
>> step( Gc, 15 )
```



5.2 Allmän insignal

För att simulera linjära system med allmänna insignaler kan man använda funktionen `lsim(G,u,t)`. Indata till denna funktion är ett (eller flera) system `G`, en insignalvektor `u` och en tidsvektor `t`.

Antag exempelvis att vi vill studera reglerfelet för det återkopplade systemet ovan då referenssignalen är en ramp. Vi vet att sambandet mellan referenssignal och reglerfel ges av känslighetsfunktionen

$$E(s) = S(s)R(s)$$

där

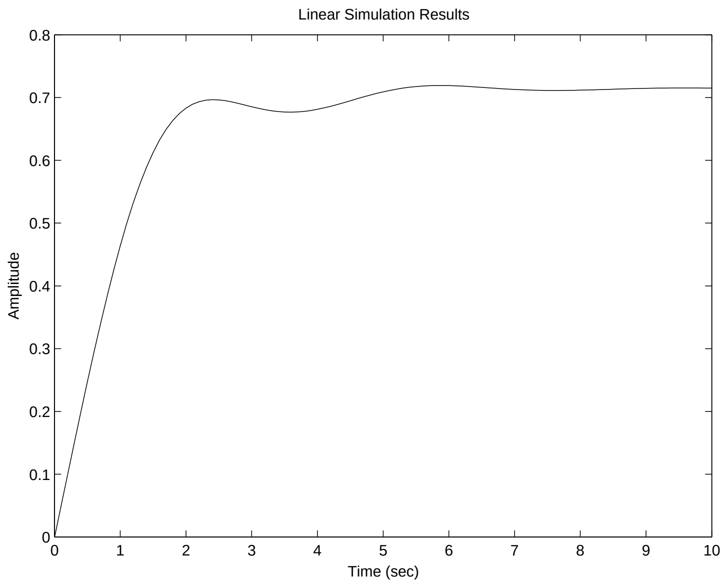
$$S(s) = \frac{1}{1 + G(s)F(s)}$$

Skapa en tidsvektor mellan 0 och 10 med steget 0.1.

```
>> t = ( 0 : 0.1 : 10 ).';
```

Simulera det återkopplade systemet då referenssignalen är en ramp med lutning 0.5. Reglerfelet går i detta fall mot 0.71. Funktionen ritar även insignalen, men den kan välja bort på menyn som nås via höger musknapp.

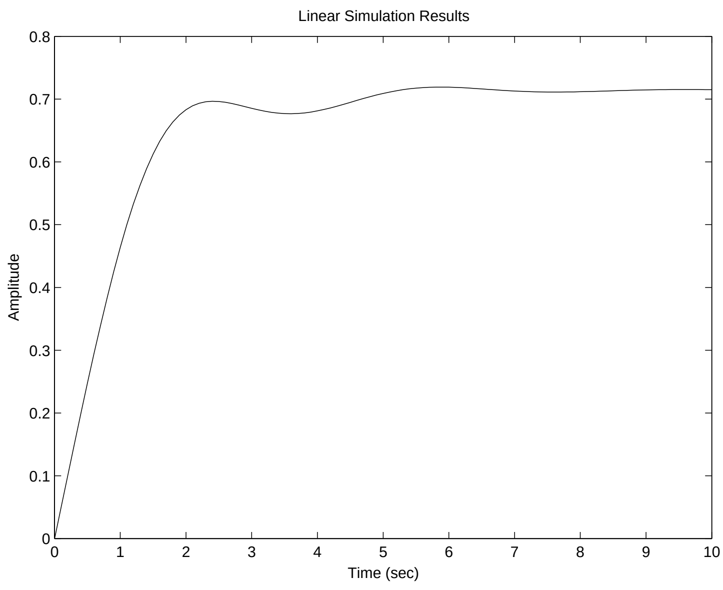
```
>> lsim( S, 0.5*t, t )
```



Vi kan naturligtvis även skapa sinus- och fyrkantsignaler eller vad vi nu kan tänkas vara intresserade av

Skapa en sinussignal för att se hur mycket signalen förstärks, vilket vi senare kan relatera till beräkningar som Bodediagram. Vi kan t.ex se vad amplituden på utsignalen blir om vi skickar in en sinusformad referenssignal med frekvensen 3 rad/s

```
>> t = 0:0.01:10;u = sin(3*t);
>> lsim( Gc, u, t )
```



---

6 Rotort

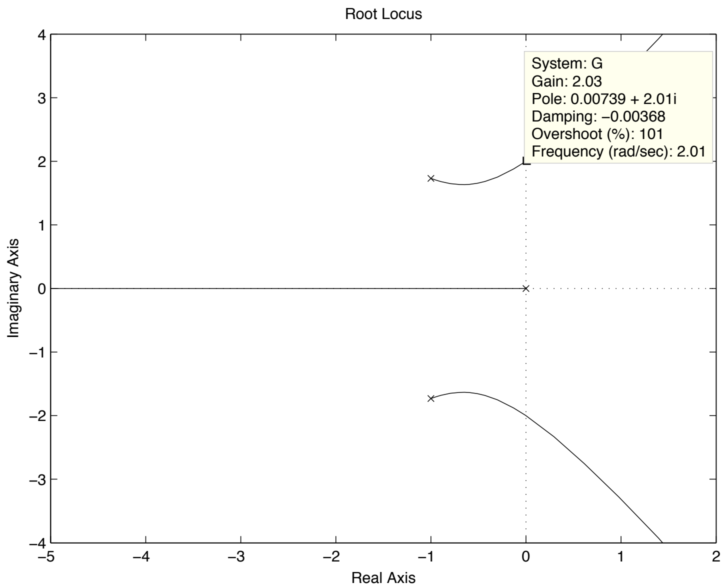
För att avgöra hur rötterna till ekvationen

$$P(s) + KQ(s) = 0$$

rör sig i komplexa talplanet då K går från noll och mot oändligheten kan man rita ekvationens rotort med funktionen `rlocus`. Indata till funktionen är en överföringsfunktion med polynomet $Q(s)$ som täljare och polynomet $P(s)$ som nämnare (rotortsanalysen motsvarar rötter i ett slutet systemet där $G(s) = Q(s)/P(s)$ styrs med en P-regulator med förstärkning K). Med höger musknapp kan man markera relevanta punkter i figuren, såsom t ex då rotorten passerar imaginäraxeln.

Rotort för $P(s) + KQ(s) = 0$. Markera där en av rötterna passerar imaginäraxeln.

```
>> rlocus( Q/P )
```



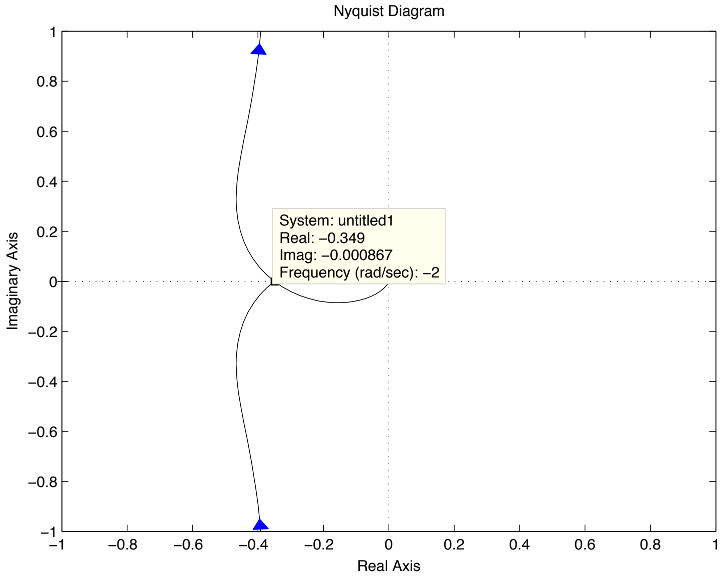
För att t ex kontrollera för vilken förstärkning polerna har viss dämpning kan man med höger musknapp lägga in ett nät vilket markerar polplaceringar med samma avstånd till origo respektive samma dämpning.

7 Nyquistdiagram (ingår ej i alla kurser)

Nyquistkurvor för en eller flera överföringsfunktioner ritas med funktionen `nyquist`. Eftersom funktionen `nyquist` graderar axlarna automatiskt kan diagrammet ibland bli svårläst. Läsbarheten kan förbättras genom att man själv väljer axlarnas gradering med funktionen `axis`. Man kan få ut mycket information ur figuren genom att använda vänster respektive höger musknapp. Med vänster musknapp kan man t ex markera en punkt på kurvan och få ut motsvarande värde på ω samt nyquistkurvans värde i denna frekvens. Med höger musknapp får man en meny med olika operationer som kan göras med figuren.

Rita nyquistkurvan för det öppna systemet då systemet $G(s)$ styrs med en proportionell återkoppling med förstärkning $K_P = 0.7$. Justera axlarnas gradering och markera punkten där nyquistkurvan passerar negativa delen av reella axeln.

```
>> nyquist( G * F )
>> axis([ -1 1 -1 1 ])
```



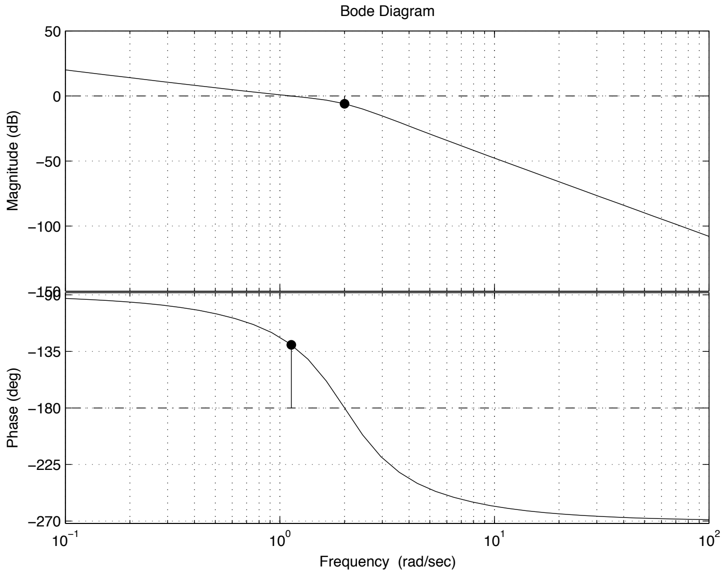
---

8 Bodediagram

Bodediagram för en eller flera överföringsfunktioner ritas med funktionen `bode`. Även i detta fall kan man läsa av punkter i figuren genom att markera med vänster musknapp. Med höger knapp får man en meny där man t ex kan välja att markera frekvenserna där stabilitetsmarginalerna läses av.

Beräkna frekvensfunktionen för systemet `G` och rita upp den i ett bodediagram. Notera att amplitudkurvan graderas i decibel. Använd höger musknapp för att lägga in rutnät i figuren samt markera var fas- och amplitudskärfrekvenserna ligger. Det går även att byta till andra skalor i menyer som kommer upp i högerknapp. Om du bara vill rita amplitudkurvan kan du använda `bodemag`.

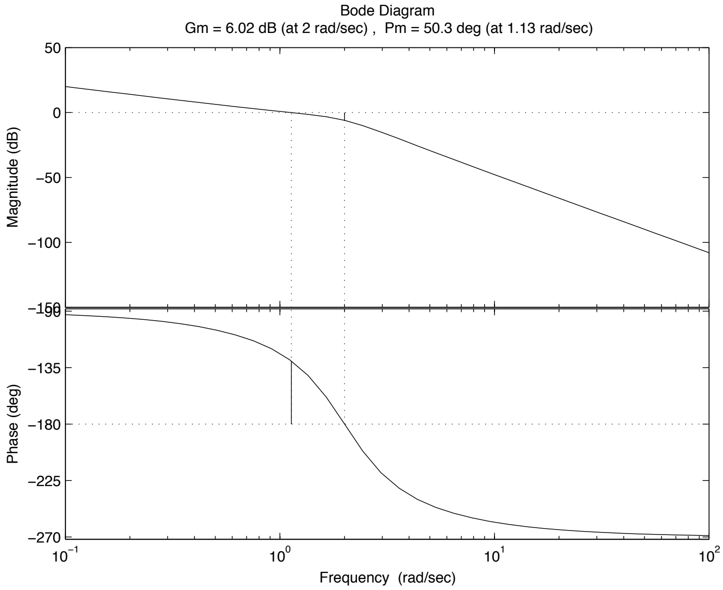
```
>> bode( G )
```



För att bestämma skärfrekvenser samt fas- och amplitudmarginal kan man även använda funktionen `margin`, vilken förutom att rita upp amplitud- och faskurvorna även skriver ut dessa värden. `Gm` och `Pm` betecknar amplitud- respektive fasmarginal.

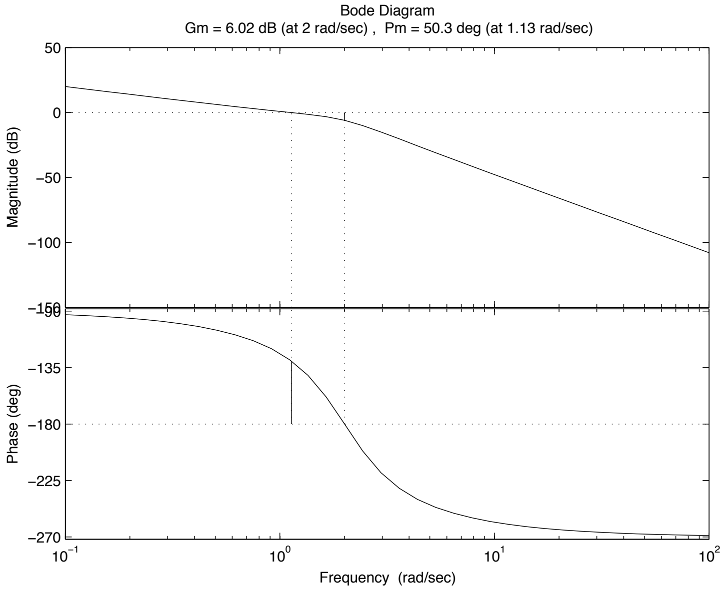
Beräkna frekvensfunktionen för systemet G och rita upp den i ett bodediagram.

```
>> margin( G )
```



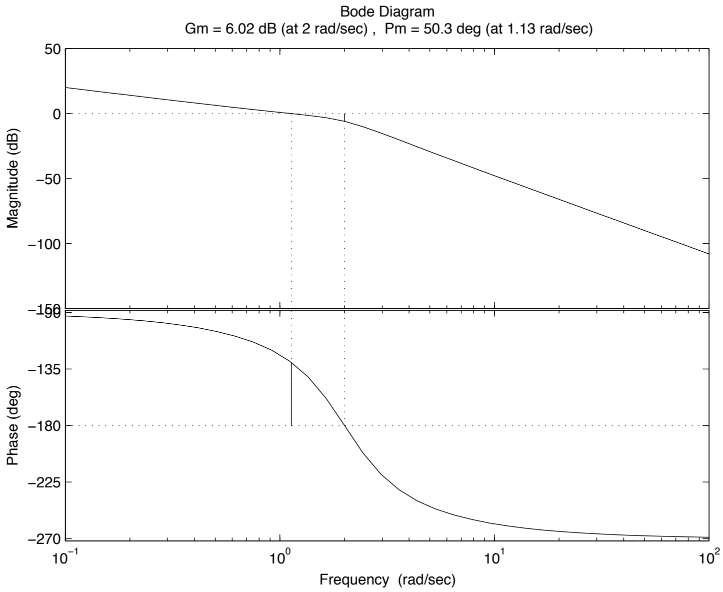
Amplitudmarginalen och fasskärfrekvensen sägs vara 6.02dB resp 2 rad/s. Detta betyder alltså att frekvenssvaret ska ha en fas på -180° i 2 rad/s och således vara ett negativt reellt tal där. Vi kan verifiera siffrorna genom att använda kommandot **freqresp** som beräknar frekvensfunktionen för ett LTI-objekt i en given frekvens. I praktiken vill vi dessutom inte använda dB

```
>> Am = 10^(6.02/20)

Am =

1.9999
>> wp = 2;
>> freqresp(G,wp)
ans =

-0.5000
>> 1/abs(freqresp(G,wp))
ans =

2
```



Skärfrekvensen sägs vara 1.13 rad/s och fasmarginal 50.3°. Ta tillfället i akt och markera den komplexa punkten $G(i\omega_c)$ i komplexa talplanet och verifiera geometriskt vad det betyder att dess vinkel (fas) är $-129.7°$, och att avståndet till -180^{circ} $+50.3°$.

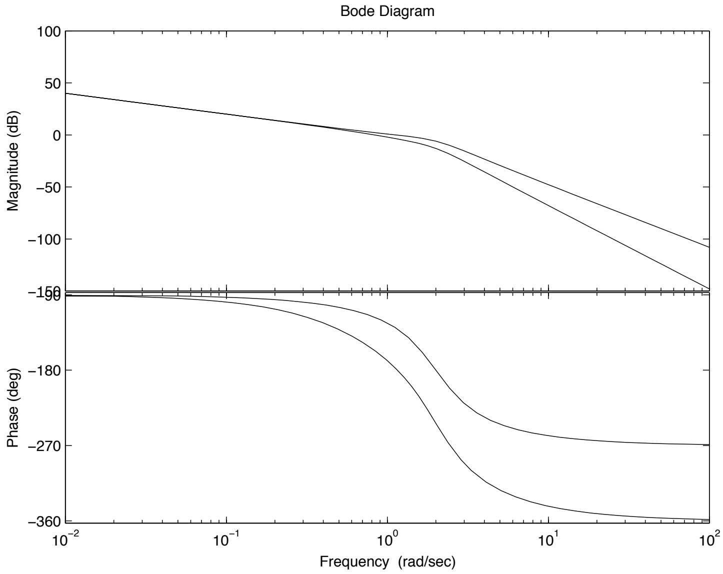
```
>> freqresp(G,wc)

ans =

-0.6388 - 0.7697i

>> abs(freqresp(G,wc))

ans =

1.0003

>> phase(freqresp(G,wc))*180/pi

ans =

-129.6905

>> phase(freqresp(G,wc))*180/pi--180

ans =

50.3095
>> plot(freqresp(G,wc),'*');grid on;axis([-1 1 -1 1])
```



För att t ex kunna göra jämförelser mellan två frekvensfunktioner kan dessa ritas i samma diagram.

Beräkna frekvensfunktionerna för systemen G och P definierade tidigare.

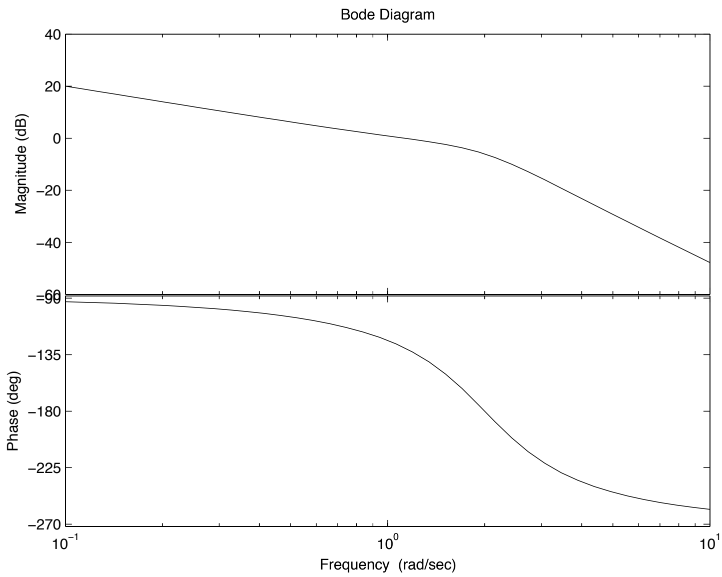
```
>> bode( G, P )
```



Skalan på frekvensaxeln kan väljas genom att som sista argument i funktionsanropet ange största och minsta frekvensvärdet mellan krullparenteser .

Beräkna frekvensfunktionen för systemet G från 0.1 till 10 rad/s och rita upp den i ett bodediagram.

```
>> bode( G, { 0.1, 10 } )
```



Alternativt kan man skicka med en lista med frekvenser som man vill beräkna frekvensfunktionen i. Eftersom man ritar i logaritmiska axlar vill man typiskt ha en logaritmisk spridning på värdena vilket kan genereras med `logspace` (alternativet är `linspace`)

Skapa 1000 frekvenser mellan 10^{-1} och 10^2 och rita upp frekvenssvaret i dessa frekvenser

```
>> bode( G, logspace(-1,2,1000))
```

9 Tillståndsbeskrivning

Control System Toolbox kan naturligtvis även hantera system på tillståndsform

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

För att skapa ett system på denna form används funktionen `ss`, med vilken man kan skapa ett system på tillståndsform från början eller konvertera ett system från överföringsfunktionsform.

Överför systemet `G` till tillståndsform.

```
>> G = ss( G )

a =
       x1  x2  x3
   x1  -2  -2   0
   x2   2   0   0
   x3   0   1   0

b =
       u1
   x1   2
   x2   0
   x3   0

c =
       x1  x2  x3
   y1   0   0   1

d =
       u1
   y1   0

Continuous-time model.
```

Man kan även gå åt andra hållet

```
>> G = tf(G)

G =

   4
 -----------------
 s^3 + 2 s^2 + 4 s

Continuous-time transfer function.
```

Matriserna A, B, C och D i tillståndsbeskrivningen ingår nu i datastrukturen `G`. För att komma åt matriserna kan man referera till dem direkt genom att skriva `G.a`, `G.b` etc.

Beräkna egenvärdena till matrisen A i tillståndsmodellen

```
>> G = ss(G);
>> eig( G.a )

ans =

        0
  -1.0000 + 1.7321i
  -1.0000 - 1.7321i
```

Tidigare kommandon som vi lärt oss fungerar naturligtvis fortfarande. Hurevida vi har systemet i tillståndsform eller överföringsfunktionsform handlar främst om representation, det är inte något som påverkar systemets egenskaper.

```
>> pole(G)
ans =

 0.0000 + 0.0000i
 -1.0000 + 1.7321i
 -1.0000 - 1.7321i
```

Polplacering tillståndsåterkoppling på formen

$$u(t) = -Lx(t) + r(t)$$

kan göras med funktionen `place` (och i special-fall med funktionen `acker`).

Bestäm en tillståndsåterkoppling som placerar det återkopplade systemets poler i närheten av -2. Lägger man alla polerna exakt i -2 kan återkopplingen inte beräknas med hjälp av `place` (det är en begränsning i kommandot), så vi spider ut polerna en aning. För att få alla polerna exakt i -2 kan `acker` användas, men den funktionen har ofta så dåliga numeriska egenskaper att den bör undvikas till förmån för `place`.

```
>> L = place( G.a, G.b, ...
              -2 * ( 1 + [ -0.01 0 0.01 ] ) )

L =

    2.0000    1.9999    1.9998
```

Det återkopplade systemet

$$\dot{x}(t) = (A - BL)x(t) + Br(t)$$
$$y(t) = Cx(t)$$

kan nu skapas t ex med funktionen `ss`.

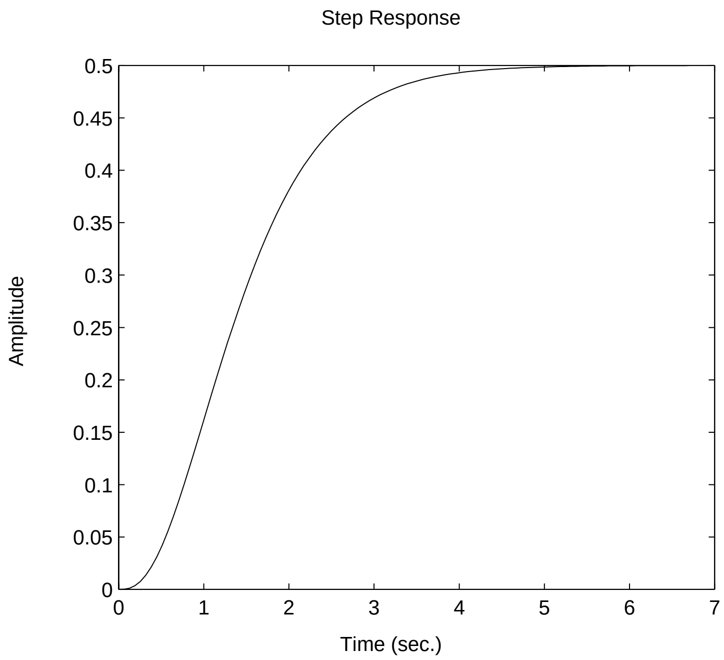
Generera tillståndsbeskrivningen för det återkopplade systemet. Kontrollera att polerna placerats på önskat sätt. Notera att detta är ett linjärt system vars insignal är r.

```
>> Gc = ss( G.a - G.b * L, G.b, G.c, 0 );
>> eig( Gc.a )

ans =

   -2.0200
   -2.0000
   -1.9800
```

Det återkopplade systemets stegsvar kan nu beräknas och ritas upp med funktionen `step` som vanligt.

Beräkna och rita upp det återkopplade systemets stegsvar.

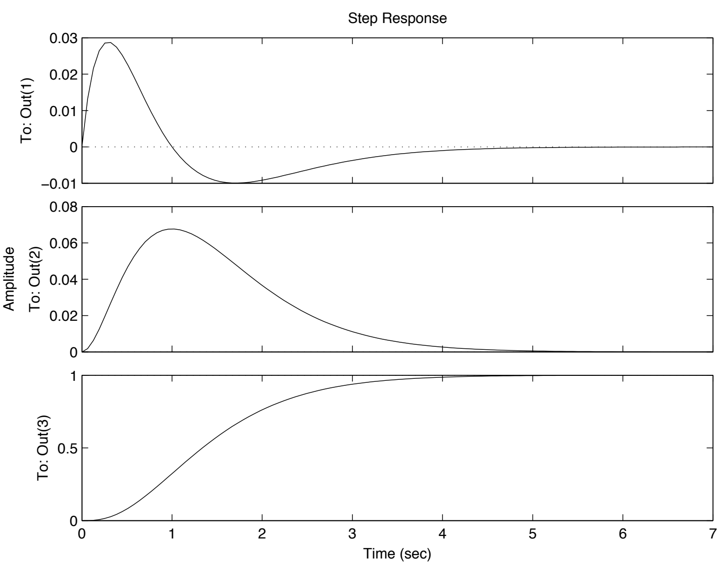
```
>> step( Gc )
```



På detta sätt ser vi endast den utsignal som definieras av vektorn C. Vill vi studera samtliga tillstånd kan detta göras genom att låta C vara en enhetsmatris med dimension lika med systemets ordningstal.

Skapa det återkopplade systemet på nytt, men med samtliga tre tillstånd som utsignaler, $y = Ix + 0r$. Notera att vi explicit beskriver storleken på nollmatrisen i koden för att säkerställa att vi förstår algebran. För att matematiken ska kunna gpå ihop måste I vara en enhetsmatris med samma dimension som antalet tillstånd x samtidigt som 0 måste vara en matris med samma höjd som y (dvs x) och bredd som antalet referenssignaler r vilket är 1.

```
>> Gc = ss( G.a - G.b * L, G.b, eye(3), zeros(3,1) );
>> step( Gc )
```



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10 Sammanfattning av kommandon

10.1 Användbara kommandon i Control System Toolbox

`tf`	System på överföringsfunktionsform
`ss`	System på tillståndsform
`pole`	Poler
`step`	Stegsvar
`tzero`	Nollställen
`feedback`	Återkoppling
`nyquist`	Nyquistdiagram
`bode`	Bodediagram
`bodemag`	Bodediagrammets amplitudkurva
`margin`	Bodediagram och stabilitetsmarginaler
`freqresp`	Frekvenssvar i specifik frekvens
`rlocus`	Rotort
`lsim`	Simulering med godtycklig insignal
`place`	Polplacerande tillståndsåterkoppling
`ctrb`	Styrbarhetsmatris
`obsv`	Observerbarhetsmatris
`pzmap`	Pol-nollställediagram
`minreal`	Förkortning av gemensamma faktorer

10.2 Användbara MATLAB-kommandon

`abs`	Absolutbelopp
`phase`	Fas/argument
`eig`	Egenvärden
`det`	Determinant
`diag`	Diagonalmatris
`imag`	Imaginärdel
`inv`	Matrisinvers
`real`	Realdel
`imag`	Imaginärdel
`roots`	Rötter till polynom
`grid`	Nät i figurer
`axis`	Ställ in koordinatsystem i figur
`hold`	Frysning av figur
`loglog`	Diagram i log-log skala
`plot`	Diagram i linjär skala
`cd`	Byte av bibliotek
`dir`	Listning av bibliotek
`clear`	Radering av variabler och funktioner i arbetsminnet
`load`	Inläsning av variabler från fil
`save`	Lagring av variabler på fil
`who`	Listning av variabler i arbetsminnet
`helpdesk`	Startar HTML-baserad hjälpfunktion