Optimization Problems

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March 26, 2024

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Least Squares (LS) Problem

$$\text{minimize} \frac{1}{2} \sum_{k=1}^{m} f_k(x)^2 \tag{1}$$

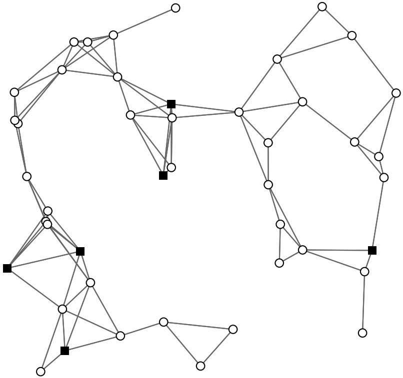
with variable $x \in \mathbf{R}^n$, where $f_k : \mathbf{R}^n \to \mathbf{R}$.

Compactly:

$$\text{minimize} \frac{1}{2} \|f(x)\|_2^2$$

with variable $x \in \mathbf{R}^n$, where $\| \cdot \|_2$ is the Euclidian norm and where $f(x) = (f_1(x), \ldots, f_m(x))$.

Localization Problem



Let $a_k \in \mathbf{R}^D$ for $k \in \mathbf{N}_m$ be known *anchor positions*.

Let x_i for $i \in \mathbf{N}_n$ be *unknown positions* to be determined.

Localization Problem ctd.

Range measurements $r_{i,j} \in \mathbf{R}$ for $(i,j) \in \mathcal{E} \subset \mathbf{N}_n \times \mathbf{N}_n$ and $v_{i,k}$ for $(i,k) \in \mathcal{E}_a \subset \mathbf{N}_n \times \mathbf{N}_m$.

LS problem:

$$\text{minimize} \sum_{(i,j)\in\mathcal{E}} \left(\|x_i - x_j\|_2 - r_{i,j}\right)^2 + \sum_{(i,j)\in\mathcal{E}_a} \left(\|x_i - a_k\|_2 - v_{i,k}\right)^2$$

with variables $(x_1, \ldots, x_n)$.

Linear LS

Linear LS: $f_k(x) = a_k^T x - b_k$, where $a_k \in \mathbf{R}^n$ and $b_k \in \mathbf{R}$.

For the linear case $f(x) = Ax - b$, where

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbf{R}^{m \times n}; \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Optimality Conditions

Necessary conditions for optimality of nonlinear LS problem:

$$\frac{\partial}{\partial x}\frac{1}{2}\sum_{k=1}^{m} f_k(x)^2 = \sum_{k=1}^{m} f_k(x)\frac{\partial f_k(x)}{\partial x} = 0 \qquad (2)$$

For linear case optimality conditions are:

$$\sum_{k=1}^{m}\left(a_k^T x - b_k\right) a_k = 0 \qquad (3)$$

or equivalently

$$A^T A x = A^T b$$

called *Normal equation*.

Constrained LS Problem

For linear case:

$$\text{minimize } \frac{1}{2}\|Ax - b\|_2^2 \tag{4}$$

$$\text{subject to } Bx = c \tag{5}$$

with variable $x \in \mathbf{R}^n$, where $B \in \mathbf{R}^{p \times n}$.

Optimality conditions (KKT equation):

$$\begin{bmatrix} A^T A & B^T \\ B & \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T b \\ c \end{bmatrix} \tag{6}$$

where $\lambda \in \mathbf{R}^p$

Constraint qualification is satisfied if B full row rank.

Unique solution if and only if $A^T A$ is postive definite on the nullspace of B.

Quadratic Program (QP)

$$\begin{aligned} \text{minimize} \ & \frac{1}{2}x^T Qx + r^T x & (7) \\ \text{subject to} \ & Bx = c & (8) \end{aligned}$$

with variable $x \in \mathbf{R}^n$, where $Q \in \mathbf{S}^n_+$, $r \in \mathbf{R}^n$, $B \in \mathbf{R}^{p \times n}$, and $c \in \mathbf{R}^p$.

LS problem is a QP if $Q = A^T A$ and $r = A^T b$.

QP with Affine Inequality Constraints

$$\text{minimize } \frac{1}{2}x^T Q x + r^T x \tag{9}$$

$$\text{subject to } Bx = c \tag{10}$$

$$Cx \leq d \tag{11}$$

with variable $x \in \mathbf{R}^n$, where $C \in \mathbf{R}^{q \times m}$ and $d \in \mathbf{R}^m$.

KKT optimality conditions:

$$\begin{aligned} Qx + B^T\mu + C^T\lambda &= -r \\ Bx - c &= 0 \\ Cx + s - d &= 0 \\ s_i\lambda_i &= 0, \qquad i \in \mathbf{N}_q \end{aligned}$$

together with $\lambda \geq 0$ and $s \geq 0$, where $\mu \in \mathbf{R}^p$, $\lambda \in \mathbf{R}^q$.

Special case when $Q = 0$, which is called a *Linear Program* (LP).

Example

Let $X : \mathbf{N}_n \to \mathbf{R}$ be random variable and

$$p_k = P(X = x_k), \quad x_k \in \mathbf{R}$$

Valid p satisfies $p_k \geq 0$, $k \in \mathbf{N}_n$, and $\sum_{k=1}^n p_k = \mathbf{1}^T p = 1$.

Let $f_i : \mathcal{D} \to \mathbf{R}$ for $i \in \mathbf{Z}_m$ and assume

$$Ef_i(X) \leq b_i, \qquad i \in \mathbf{N}_m$$

or equivalently

$$a_i^T p \leq b_i, \qquad i \in \mathbf{N}_m$$

where $a_i = (f_i(x_1), \ldots, f_i(x_n))$.

Example ctd.

With $c = (f_0(x_1), \ldots, f_0(x_n))$ the LP

$$\begin{aligned} \text{minimize}_{p \in \mathbf{R}^n} \; & c^T p \\ \text{subject to} \; & \mathbf{1}^T p = 1 \\ & Ap \leq b \\ & p \geq 0 \end{aligned}$$

where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}; \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

minimizes $Ef_0(X)$ subject to $Ef_i(X) \leq b_i, \; i \in \mathbf{N}_m$. The solution $p^\star$ defines the optimal probability function.

What about $\mathrm{Var} f_0(X)$?

Can be expressed as

$$\mathrm{Var} f_0(X) = Ef_0(X)^2 - (Ef_0(X))^2 = r^T p - \left(c^T p\right)^2$$

where $r = (f_0(x_1)^2, \ldots, f_0(x_n)^2)$.

Hence with $Q = 2cc^T$

$$-\mathrm{Var} f_0(X) = \frac{1}{2} p^T Q p - r^T p$$

Maximimizing the variance subject to expectation constraints is a convex QP.

Cones

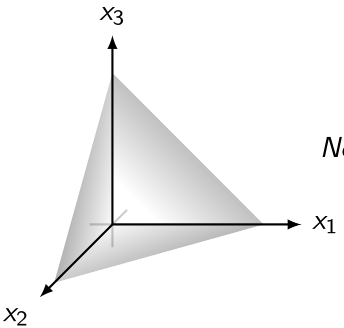
A set C is a *cone* if for any $x \in C$ and for any $\theta \geq 0$ it holds that $\theta x \in C$.

A set C is a *convex cone* if it is convex and a cone.

This holds if for any $x, y \in C$ and any $\theta_1,\ \theta_2 \geq 0$ it holds that $\theta_1 x + \theta_2 y \in C$.

A cone is *proper* if it is closed, pointed, i.e. $C \cap (-C) = \{0\}$ and has nonempty interior.

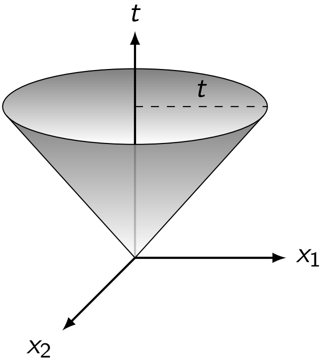
Examples of Cones



Nonnegative orthant:

$$\mathbf{R}^n_+ = \{x \mid x \geq 0\} \subseteq \mathbf{R}^n$$

Examples of Cones ctd.



Norm cone:

$$\left\{(x,t) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid \|x\| \leq t\right\} \subseteq \mathbf{R}^n$$

where $\|\cdot\| : \mathbf{R}^{n-1} \to \mathbf{R}$ is any norm.

When the norm is the Euclidian norm we get the *Second-order cone*, the *quadratic cone*, the *Lorentz cone* or the *ice-cream cone* denoted $\mathbf{Q}^n$.

Positive semidefinite cone:

$$\mathbf{S}^n_+ = \{X \in \mathbf{S}^m \mid X \succeq 0\}$$

Epigraph of Perspective of Convex Function

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function and

$$K = \{(x, y, z) |\; y > 0,\; yf(x/y) \leq z\}$$

Then $K \cup \{0, 0, 0\}$ is a convex cone.

If the interior of **dom** f is not empty and if **epi** f does not contain a straight line, then the cone is proper.

Any convex inequality constraint $f(x) \leq t$ can be rephrased as convex conic constraint $(x, 1, t) \in K$ modulo some minor modifications to make K proper.

Exponential cone when $f(x) = \exp(x)$ and $n = 1$:

$$K_{\exp} = \{(x, y, z) \in \mathbf{R}^3 |\; y > 0,\; y \exp(x/y) \leq z\} \cup \{(0, 0, 0)\}$$

Conic Optimization Problem

The following optimization problem is called a *conic optimization problem*:

$$\text{minimize } c^T x \tag{12}$$
$$\text{subject to } Ax = b \tag{13}$$
$$x \in C \tag{14}$$

where $A \in \mathbf{R}^{p \times n}$, $b \in \mathbf{R}^p$, $c \in \mathbf{R}^n$ and $C \subseteq \mathbf{R}^n$ is a proper cone.

There are special purpose solvers for conic optimization problems, e.g. MOSEK.

Dual Cone

$$C^* = \{y \mid \langle x, y\rangle \geq 0 \; \forall x \in C\}$$

where $\langle \cdot, \cdot \rangle$ is appropriate inner product.

If C proper cone, then C^* proper cone, $(C^*)^* = C$ and

$$\textbf{int}\, C^* = \{y \mid \langle x, y\rangle > 0 \; \forall x \in C, \; x \neq 0\}$$

A cone is self-dual if $C^* = C$. The cones $\mathbf{R}^n_+$, $\mathbf{Q}^n$, and $\mathbf{S}^n_+$ are self-dual.

The dual of the exponential cone K_{exp} is

$$K^*_{\text{exp}} = \{(u, v, w) \in \; \mathbf{R}_- \times \mathbf{R} \times \mathbf{R}_+ \mid \; -u \ln(-u/w) + u - v \leq 0\}$$

Duality

Lagrangian $L : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}$ for conic optimization problem:

$$L(x, \lambda, \mu) = c^T x - \lambda^T x + \mu^T (Ax - b)$$

Assume that $C \subseteq \mathbf{R}^n$. [1]

Lagrange dual function $g : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}$, where

$$g(\lambda, \mu) = \min_x L(x, \lambda, \mu)$$

where $g(\lambda, \mu) = -b^T \mu$ when $c - \lambda - A^T \mu = 0$ and $-\infty$ otherwise.

$g(\lambda, \mu) \leq p^\star$ for $\lambda \in C^\star$.

[1] For $\mathbf{S}_+^m$ we use a symmetric vectorization of $X \in \mathbf{S}^m$, i.e. $x = \mathbf{svec}\, X$, where $\mathbf{svec} : \mathbf{S}^m \to \mathbf{R}^n$ with $n = m(m+1)/2$.

Lagrange Dual Problem

$$\begin{aligned} \text{maximize} \quad & -b^T\mu && (15) \\ \text{subject to} \quad & A^T\mu + \lambda = c && (16) \\ & \lambda \in C^\star && (17) \end{aligned}$$

with variables (λ, μ). KKT conditions:

$$\begin{aligned} A^T\mu + \lambda &= c \\ Ax &= b \\ x^T\lambda &= 0 \end{aligned}$$

together with $x \in C$ and $\lambda \in C^\star$.

Necessary and sufficient for optimality if either the primal problem or the dual problem admits a strictly feasible point.

Moment Constraints

Let $X : \Omega \to \mathbf{R}$ be a random variable, Let $m_k = EX^k$, $0 \le k \le 2n$.
Let $H : \mathbf{R}^{2n+1} \to \mathbf{R}^{(n+1)\times(n+1)}$:

$$H(m_0, \dots, m_{2n}) = \begin{bmatrix} m_0 & m_1 & m_2 & \dots & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & \dots & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & \dots & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-2} & m_{2n-1} \\ m_n & m_{n+1} & m_{n+2} & \dots & m_{2n-1} & m_{2n} \end{bmatrix}$$

Then $H(m_0, \dots, m_{2n}) \in \mathcal{S}_+^{n+1}$, since

$$x^T H(m_0, \dots, m_{2n}) x = \sum_{i=0}^{n} \sum_{j=0}^{n} x_i x_j EX^{(i+j)} = E \left(\sum_{k=0}^{n} x_k X^k \right)^2 \ge 0$$

for any $x \in \mathbf{R}^{n+1}$. There is a partial converse.

Moment Constraints ctd.

Let $p : \mathbf{R} \to \mathbf{R}$ be given by $p(x) = \sum_{k=0}^{2n} c_k x^k$. Then

$$E(p(X)) = \sum_{k=0}^{2n} c_k m_k$$

Hence optimization problem

$$\begin{aligned} &\text{minimize } E(p(X)) \\ &\text{subject to } l_k \leq EX^k \leq u_k, \quad 1 \leq k \leq 2n \end{aligned}$$

over all random variables can be solved as the conic program

$$\begin{aligned} &\text{minimize } \sum_{k=0}^{2n} c_k m_k \\ &\text{subject to } l_k \leq m_k \leq u_k, \quad 1 \leq k \leq 2n \\ &\qquad H(1, m_1, \ldots, m_{2n}) \in \mathcal{S}_+^{n+1} \end{aligned}$$

with variables $(m_1, \ldots, m_{2n})$. Notice that we do not have to know the pdf of the random variable in the problem formulation above.

Rank -Constrained Optimization

Let $X, Z \in \mathbf{R}^{m\times n}$, and consider:

$$\text{minimize } \frac{1}{2}\|Z - X\| \tag{18}$$

$$\text{subject to } \mathbf{Rank}\, Z \leq r \tag{19}$$

with variable Z, where $r \leq q = \min(m, n)$ and where $\|\cdot\|$ denotes the matrix norm induced by the Euclidian norm.

Let $X = UDV^T$ be an SVD with diagonal elements d_i of D in decreasing order.

$Z = U_1 D_1 V_1^T$, where U_1 and V_1 contains the first r columns of U and V, respectively, and where $D = D_1 \oplus D_2$ is an optimal Z.

Rank Optimization

Rank-constrained optimization is closely related to the *rank optimization* problem:

$$\text{minimize} \quad \mathbf{Rank}\, Z \tag{20}$$

$$\text{subject to} \quad \frac{1}{2}\|Z - X\| \leq \varepsilon \tag{21}$$

with variable Z, where $X, Z \in \mathbf{R}^{m \times n}$, and where $\varepsilon \in \mathbf{R}_+$.

The solution for one of them can readily be obtained from the other by doing bi-sectioning on r or ε, respectively.

General Rank Minimization Problem

$$\text{minimize} \quad \textbf{Rank}\, Z \tag{22}$$
$$\text{subject to} \quad Z \in \mathcal{C} \tag{23}$$

with variable Z, where $\mathcal{C} \subset \mathbf{R}^{m \times n}$ typically a convex set.

The overall problem is however not convex.

Heuristic Rank Optimization

Assume that Z is square and positive semidefinite.

Heuristic optimization problem:

$$\begin{align} \text{minimize} \;\; & \mathbf{tr}\, Z \tag{24} \\ \text{subject to} \;\; & Z \in \mathcal{C} \tag{25} \\ & \tag{26} \end{align}$$

with variable Z, where $Z \in \mathbf{S}_+^m$.

This follows from the fact that $\mathbf{tr}\, Z = \sum_{i=1}^m \lambda_i(Z) = \|\lambda(Z)\|_1$, where $\lambda(Z)$ is the vector of eigenvalues of Z. Since minimizing the ℓ_1-norm is known to result in a sparse vector, we will obtain several eigenvalues that are zero, and hence a low-rank matrix Z.

General Z

For general $Z \in \mathbf{R}^{m \times n}$ it holds that $\mathbf{Rank}\, Z \leq r$ if and only if there exist matrices such that $\mathbf{Rank}\, X + \mathbf{Rank}\, Y \leq 2r$ and

$$\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} \succeq 0$$

Because of this we consider the heuristic

$$\begin{aligned} \text{minimize}_{X,Y,Z} \; & \frac{1}{2} \mathbf{tr}\, \mathbf{bdiag}(X, Y) && (27) \\ \text{subject to} \; & \begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} \succeq 0 && (28) \\ & Z \in \mathcal{C} && (29) \end{aligned}$$

Nuclear Norm Interpretation

Equivalent problem:

$$\text{minimize } \|Z\|_* \tag{30}$$
$$\text{subject to } Z \in \mathcal{C} \tag{31}$$

with variable Z, where $\|\cdot\|_* : \mathbf{R}^{m\times n} \to \mathbf{R}_+$ is the *nuclear norm* or *Kay-Fan n-norm* of Z.

This norm is the dual of the induced norm and given by

$$\|Z\|_* = \sum_{i=1}^{q} \sigma_i(Z)$$

where $q = \min(m, n)$, and where $\sigma_i(Z)$ are the singular values of Z.

Convex Envelope

Can be shown that $\|Z\|_*$ is the *convex envelope* of **Rank** Z on the set $\mathcal{D} = \left\{ Z \in \mathbf{R}^{m \times n} \mid \|Z\| \leq 1 \right\}$.

That means that $\|Z\|_*$ is the largest convex function such that $\|Z\|_* \leq \textbf{Rank}\, Z$ for all $Z \in \mathcal{D}$.

Hence the nuclear norm heuristic replaces the nonconvex rank-function with the tightest lower convex bound.

However, it is only tight in case we consider the constraint set $\mathcal{D}$, but in many cases it is a good heuristic also without this constraint.

Log-det Heuristic

For a low rank matrix $Z \in \mathbf{S}_+^m$ and for small values of $\delta > 0$ we have $\ln\det(Z + \delta I)$ negative with a large absolute value.

Heuristic for rank minimization when $Z \in \mathbf{S}_+^m$:

$$\begin{aligned} \text{minimize } & \ln\det(Z + \delta I) && (32) \\ \text{subject to } & Z \in \mathcal{C} && (33) \end{aligned}$$

with variable Z. Notice that $\ln\det(Z + \delta I)$ is not a convex function of Z but instead concave.

General Z

For general $Z \in \mathbf{R}^{m \times n}$ we just as before embed the problem and obtain the heuristic

$$\begin{aligned} \text{minimize}_{X,Y,Z} \quad & \ln\det(\mathbf{bdiag}(X, Y) + \delta I) && (34) \\ \text{subject to} \quad & \begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} \succeq 0 && (35) \\ & Z \in \mathcal{C} && (36) \end{aligned}$$

with variable (X, Y, Z), where $X \in \mathbf{S}^m$ and $Y \in \mathbf{S}^n$.

Minimal State-Space Realization

Let $H : \mathbf{R}^{n-1} \to \mathbf{R}^{n\times n}$ be defined by

$$H(h_{n+1},\ldots,m_{2n-1}) = \begin{bmatrix} h_1 & h_2 & h_3 & \ldots & h_{n-1} & h_n \\ h_2 & h_3 & h_4 & \ldots & h_n & h_{n+1} \\ h_3 & h_4 & h_5 & \ldots & h_{n+1} & h_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ h_{n-1} & h_n & h_{n+1} & \ldots & h_{2n-3} & h_{2n-2} \\ h_n & h_{n+1} & h_{n+2} & \ldots & h_{2n-2} & h_{2n-1} \end{bmatrix}$$

where $(h_1,\ldots,h_n) \in \mathbf{R}^n$ given.

The smallest r for which there are $A \in \mathbf{R}^{r\times r}$, $B \in \mathbf{R}^{r\times 1}$ and $C \in \mathbf{R}^{1\times r}$ such that $h_k = CA^{k-1}B$ for $1 \le k \le n$ is given by

$$r = \min_{h_{n+1},\ldots,h_{2n-1}} \mathbf{Rank}\, H(h_{n+1},\ldots,m_{2n-1})$$

Hence (A,B,C) provides a minimal state-space realization of the partial impulse response $(h_1,\ldots,h_n)$.